

# FANO THREEFOLDS OF GENUS 6

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ABSTRACT. Ideas and methods of *Clemens C.H.*, *Griffiths Ph.* *The intermediate Jacobian of a cubic threefold* are applied to a Fano threefold  $X$  of genus 6 — intersection of  $G(2, 5) \subset P^9$  with  $P^7$  and a quadric. We prove:

1. The Fano surface  $F(X)$  of  $X$  is smooth and irreducible. Hodge numbers and some other invariants of  $F(X)$  are calculated.
2. Tangent bundle theorem for  $X$ , and its geometric interpretation. It is shown that  $F(X)$  defines  $X$  uniquely.
3. The Abel - Jacobi map  $\Phi : \text{Alb } F(X) \rightarrow J^3(X)$  is an isogeny.
4. As a necessary step of calculation of  $h^{1,0}(F(X))$  we describe a special intersection of 3 quadrics in  $P^6$  (having 1 double point) whose Hesse curve is a smooth plane curve of degree 6.
5.  $\text{im } \Phi(F(X)) \subset J^3(X)$  is algebraically equivalent to  $\frac{2\Theta^8}{8!}$  where  $\Theta \subset J^3(X)$  is a Poincaré divisor (a sketch of the proof).

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## 0. Introduction.

The main object of the present paper is a Fano threefold  $X$  of genus 6 of the principal series which is the intersection of the Grassmann variety  $G(2, 5) \subset P^9$  with  $P^7$  and a quadric. The surface that parametrizes conics on  $X$  is called its Fano surface and is denoted by  $F(X)$ . The middle Jacobian of  $X$  is denoted by  $J^3(X)$ . We investigate the Abel - Jacobi map  $\Phi : F(X) \rightarrow J^3(X)$ . This investigation is the main component of solution of problems of rationality and of global Torelli theorem for  $X$ .

Main results obtained in the present paper are the following:

1.  $F(X)$  is smooth and irreducible. Hodge numbers and some other invariants of  $F(X)$  are calculated; it is shown that they coincide with the corresponding invariants of the two-sheeted covering of  $P^3$  ramified at a quartic.
2. Tangent bundle theorem for  $X$  is proved, and its geometric interpretation is given. It is shown that  $F(X)$  defines  $X$  uniquely (under some restriction imposed on  $X$ , see (7.1)).
3. The Abel - Jacobi map  $\Phi : \text{Alb } F(X) \rightarrow J^3(X)$  is an isogeny.
4. As a necessary step of calculation of  $h^{1,0}(F(X))$  we describe a special intersection of 3 quadrics in  $P^6$  whose Hesse curve is a smooth plane curve of degree 6 (instead of 7 in the general case) and main theorems concerning a general intersection of 3 quadrics are proved for this case.
5.  $\text{im } \Phi(F(X)) \subset J^3(X)$  is algebraically equivalent to  $\frac{2\Theta^8}{8!}$  where  $\Theta \subset J^3(X)$  is a Poincaré divisor (a sketch of the proof).

Now we give a more detailed description of the present paper. Sections 1 and 2 are of the survey nature. Section 1 contains a definition and the main properties of intermediate Jacobian of a variety, and of an Abel - Jacobi map. Section 2 contains a definition of Fano threefolds, a brief description of methods and results of proofs of their non-rationality. Main steps of proofs of non-rationality of a cubic threefold and of a two-sheeted covering of  $P^3$  ramified at a quartic (according Clemens - Griffiths) are given.

In Sections 3 and 4 we find some simple properties of  $F(X)$ , calculate some of its invariants and prove the tangent bundle theorem. Namely, let  $V$  be a vector space of dimension 5,  $G = G(2, V)$  a Grassmann variety, it is included into  $P(\lambda^2(V))$  via Plücker embedding. Let  $H_1, H_2$  be hyperplanes in  $P(\lambda^2(V))$  and  $\Omega$  a quadric in  $P(\lambda^2(V))$  such that  $X = G \cap H_1 \cap H_2 \cap \Omega$  is a smooth threefold.  $X$  is a Fano threefold of the principal series that will be studied in the present paper. We consider only those  $X$  that satisfy some conditions (3.6), (3.7) (which are fulfilled for a generic  $X$ ), although most likely all results of the paper are true for all  $X$ . Fano surface  $F_c = F_c(X)$  is a set of conics on  $X$ .

Main results of Section 3 are the following:

**Proposition 0.1.**  $F_c$  is a smooth surface containing a distinguished point  $c_\Omega$  and a distinguished straight line  $l_2$  ( $c_\Omega \notin l_2$ ). Let  $r_F : F \rightarrow F_c$  be the blowing up of  $F_c$  at  $c_\Omega$  and  $l_1 = r_F^{-1}(c_\Omega)$ . There exists an involution  $i_F$  on  $F$  without stable points such that  $i_F|_{l_1} : l_1 \rightarrow l_2$  is an isomorphism. We denote the two-sheeted covering that corresponds to  $i_F$  by  $p_F : F \rightarrow F_0$ , and we denote  $l = p_F(l_1)$ .  $l_1$  and  $l_2$  are exceptional lines on  $F$ , i.e. there are blowings down maps  $r : F \rightarrow F_m$  (resp.

$r_0 : F_0 \rightarrow F_{0m}$ ) that send lines  $l_1 \cup l_2$  (resp.  $l$ ) to points  $c_\Omega \cup c'_\Omega$  (resp.  $c_{0\Omega}$ ), as well as an involution map  $i_{F_m} : F_m \rightarrow F_m$  and the corresponding two-sheeted covering  $p_{F_m} : F_m \rightarrow F_{0m}$ , so we have a commutative diagram:

$$\begin{array}{ccccc}
& & F & & F \\
& \swarrow r_F & \downarrow & \xrightarrow{i_F} & \downarrow \\
F_c & & F_0 & & F \\
& \searrow & \downarrow i_{F_m} & & \downarrow \\
& & F_m & & F_m \\
& & \searrow p_{F_m} & & \swarrow p_{F_m} \\
& & & F_{0m} & 
\end{array}$$

where vertical maps are respectively  $r$ ,  $r_0$ ,  $r$ .

We denote  $Z = P_{P(V^*)}(O \oplus O(1))$ , there are projections  $\eta : Z \rightarrow P(V^*)$  and  $r_Z : Z \rightarrow P^5$  where  $r_Z$  is the blowing up map of a point  $t \in P^5$ .

**Proposition 0.2.** There are canonical inclusions  $\bar{\phi} : F_0 \hookrightarrow Z$ ,  $\bar{\phi}_m : F_{0m} \hookrightarrow P^5$  making the following diagram commutative (vertical maps are  $r_0$ ,  $r_Z$ ):

$$\begin{array}{ccc}
F_0 & \xrightarrow{\bar{\phi}} & Z \\
\downarrow & & \downarrow \\
F_{0m} & \xrightarrow{\bar{\phi}_m} & P^5
\end{array}$$

such that  $\bar{\phi}_m(c_{0\Omega}) = t$ . Further,  $P^5$  can be identified with a space which is dual to the space of quadrics in  $P^7$  which contain  $X$ . The point  $t \in P^5$  corresponds to a hyperplane of quadrics which contain  $G \cap H_1 \cap H_2$ .

**Proposition 0.3.** There exists a locally free sheaf  $M$  of rank 4 on  $P(V^*)$  and a bundle of quadrics  $O_\eta(-1) \rightarrow S^2(\eta^*(M^*))$  on  $Z$  such that  $F_0 \hookrightarrow Z$  is its rank 2 determinantal variety.

**Proposition 0.4.** Let  $\sigma_m \in \text{Pic } F_{0m}$  and  $\sigma = r^*(\sigma_m) \in \text{Pic } F_0$  are sheaves of order 2 that correspond to two-sheeted coverings  $p_{F_m}, p_F$  respectively. There is an explicit formula on  $F_0$  for  $\sigma$  (see (3.8)). The canonical sheaf of  $F_{0m}$  (resp.  $F_m$ ) is  $\omega_{F_{0m}} = \bar{\phi}_m^*(O_{P^5}(3)) \otimes \sigma_m$  (resp.  $\omega_{F_m} = p_{F_m}^*(\bar{\phi}_m^*(O_{P^5}(3)))$ ).

**Proposition 0.5.**  $\deg \bar{\phi}_m(F_{0m}) = 40$ ,  $c_1(\Omega_{F_m})^2 = 720$ ,  $c_2(\Omega_{F_m}) = 384$ . Taking into consideration that  $h^{0,0}(F) = h^{0,0}(F_0) = 1$  and that  $h^{1,0}(F) = 10$ ,  $h^{1,0}(F_0) = 0$  (see Theorem 0.13) we get the following table of Hodge numbers  $h^{i,j}(F_m)$ :

$$\begin{array}{ccc}
1 & 10 & 101 \\
10 & 220 & 10 \\
101 & 10 & 1
\end{array} \tag{0.6}$$

and their decomposition on  $i_{F_m}$ -invariant part  $h^{i,j}(F_m)^+ = h^{i,j}(F_{0m})$ :

$$\begin{array}{ccc}
1 & 0 & 45 \\
0 & 100 & 0 \\
45 & 0 & 1
\end{array} \tag{0.7}$$

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and on  $i_{F_m}$ -antiinvariant part  $h^{i,j}(F_m)^- = h^i(F_{0m}, \lambda^j \Omega_{F_{0m}} \otimes \sigma_m)$ :

$$\begin{array}{ccc} 0 & 10 & 56 \\ 10 & 120 & 10 \\ 56 & 10 & 0 \end{array} \quad (0.8)$$

**Proposition 0.9.** A conic  $c$  on  $X$  and a conic  $i_F(c)$  which is involutory to  $c$ , meet at 2 (possibly coinciding) points  $\gamma_1(c), \gamma_2(c)$ . We define a surface  $W = \cup_{c \in F} (\gamma_1(c) \cup \gamma_2(c))$ . Then for a generic point  $x \in W$  there exists only one (up to  $i_F$ ) conic  $c$  such that  $x \in \{\gamma_1(c), \gamma_2(c)\}$ .

**Proposition 0.10.**  $O_X(W) = O_{P^7}(21)|_X$ . There are 39 conics passing through a fixed generic point on  $X$ .

**Remark 0.11.** 0.4 – 0.8 remain true for a Fano surface  $F(X')$  of  $X'$  — a two-sheeted covering of  $P^3$  ramified at a quartic  $W$ , while 0.9 and 0.10 are not true for  $X'$ . The end of Section 3 contains a description of a special Fano threefold of genus 6 defined in [1] — a two-sheeted covering of a section of  $G(2, 5)$  by 3 hyperplanes ramified at the intersection with a quadric — whose properties are similar to the ones of  $X'$ .

Section 4 of the present paper contains a proof of the tangent bundle theorem for  $F(X)$ . Let us give the necessary definitions. There exists a locally free sheaf  $\tau_{2,M}$  of rank 2 on  $F$  such that for  $c \in F$   $P(\tau_{2,M}(c))$  is the linear envelope of  $\gamma_1(c), \gamma_2(c)$ . Let  $i : l_1 \cup l_2 \rightarrow F$  be the natural inclusion.

**Theorem 0.12.** (Tangent bundle theorem). There exists an exact sequence of sheaves on  $F$ :

$$0 \rightarrow \tau_{2,M}^* \rightarrow r^*(\Omega_{F_m}) \rightarrow i_*(2O_{l_1 \cup l_2}) \rightarrow 0$$

The main result of Section 6 is the following

**Theorem 0.13.**  $h^{1,0}(F) = 10, h^{1,0}(F_0) = 0, h^{0,0}(F) = 1, h^{0,0}(F_0) = 1$ .

The method of the proof is to consider a degeneration of  $X$  in a family of Fano threefolds  $f_X : X_4 \rightarrow \Delta$  whose base  $\Delta$  is a complex analytic neighbourhood of 0. A generic fibre of  $f_X$  is a smooth Fano threefold, the special fibre  $X_0 = f_X^{-1}(0)$  has one double point.  $X_0$  is birationally equivalent to a special intersection of 3 quadrics in  $P^6$  denoted by  $X'_0$ .

Let us consider main objects related to a generic intersection of 3 quadrics in  $P^{n-1}$  (denoted by  $X_8^{n-4}$ ) for an odd  $n$  ([28], [47] for any  $n$ , [21], [50] for  $n = 7$ ). Namely, attached to  $X_8^{n-4}$  is a Hesse curve  $H_n$  which is a plane curve of degree  $n$ , and its two-sheeted non-ramified covering  $p_{H_n} : \hat{H}_n \rightarrow H_n$ . We define  $S = S(p_{H_n})$  as a connected component of the set of effective divisors of degree  $n$  on  $\hat{H}_n$  such that  $d \in S \iff O_{H_n}(p_{H_n*}(d)) = O_{P^2}(1)|_{H_n}$ . We define the Fano surface  $F(X_8^{n-4})$  as the set of  $\frac{n-5}{2}$ -dimensional quadrics on  $X_8^{n-4}$ . Then:

(1) There exists an isomorphism  $\text{Pr } \hat{H}_n/H_n \rightarrow J^{n-4}(X_8^{n-4})$  ([21]), here  $\text{Pr}$  is the Prym variety;

(2) There exists an isogeny  $\text{Alb } S \rightarrow \text{Pr } \hat{H}_n/H_n$  ([50]);

(3) There exists an isomorphism  $F(X_8^{n-4}) \rightarrow S$  ([47]).

Section 5 contains a proof of the following

**Theorem 0.14.** An analog of a Hesse curve for  $X'_0$  is a smooth plane curve  $H_6$  of degree 6.  $\text{Sing}(F(X_0))$  is a double curve which is isomorphic to  $H_6$ . Analogs of the above (1) - (3) are true for  $X'_0$  after replacement of  $F(X_0)$  and  $X'_0$  by their desingularizations  $\widetilde{F(X_0)}$  and  $\tilde{X}'_0$  respectively. Particularly,  $h^{1,0}(\widetilde{F(X_0)}) = \dim \text{Pr}(\hat{H}^6/H^6) = 9$ .

To finish a proof of Theorem 0.13, we use a Clemens - Schmidt exact sequence ([40]) for a family  $f_F : F_3 \rightarrow \Delta$  whose fibres are Fano surfaces of fibres of  $f_X$ .

The next result of Section 6 is

**Theorem 0.15.** The Abel - Jacobi map  $\Phi : \text{Alb } F(X) \rightarrow J^3(X)$  is an isogeny.

Finally, a sketch of a proof of the following theorem is given:

**Theorem 0.16.**  $\Phi(F) \subset J^3(X)$  is algebraically equivalent to  $\frac{2\Theta^8}{8!}$  where  $\Theta \subset J^3(X)$  is a Poincaré divisor.

The idea of the proof is the following. A result of Beauville ([28]) implies that an analog of this result is true for the image of  $S(p_{H_6})$  in  $\text{Pr } \hat{H}^6/H^6$  (i.e. that it is equivalent to  $\frac{2\Theta^7}{7!}$ ). Investigation of topology of degeneration of abelian varieties given in [25], [31], permits to deduce (0.16) from this property.

Section 7 contains proofs of 2 theorems:

**Theorem 0.17.** (Geometric interpretation of the tangent bundle theorem). Let us denote  $V_{10} = H^0(F, \Omega_F)^*$ , and let  $B_{10} : F_m \rightarrow G(2, V_{10})$  be a map defined by  $\Omega_{F_m}$ . Let  $B_8 : F \rightarrow G(2, V_8)$  be a map defined as follows:  $B_8(f) = \langle \gamma_1(f), \gamma_2(f) \rangle$  (here  $V_8 = H_1 \cap H_2 \subset \lambda^2(V)$ ). Then there exists a natural projection  $p_{10,8} : V_{10} \rightarrow V_8$  such that

$$B_8 = G(2, p_{10,8}) \circ B_{10} \circ r$$

(recall that  $r : F \rightarrow F_m$  is a blowing down map). If  $B_{10}$  is regular at  $c_{0\Omega} \in F_m$  then  $p_{10,8}$  is a projection from a plane  $B_{10}(c_{0\Omega}) \subset V_{10}$ .

**Theorem 0.18.** Let  $X$  satisfies the property:  $B_{10}$  is regular at  $c_\Omega \in F_m$ . Then, if for another Fano threefold  $X'$  of the same type we have  $F_c(X) = F_c(X')$  then  $X = X'$ .

The proof of this theorem is based on the above geometric interpretation of the tangent bundle theorem that permits to recover uniquely a map  $B_8 : F \rightarrow G(2, V_8)$  by a given  $F_c(X)$ , and then to recover uniquely  $X$  by  $B_8$ .

## Section 1. Intermediate Jacobian of algebraic varieties and Abel - Jacobi map.

Recall that this section is of survey nature. Let  $Y$  be a smooth  $n$ -dimensional algebraic variety over  $\mathbb{C}$ . We consider the Hodge decomposition of its Betti cohomology of an odd dimension  $k$ :

$$H^k(Y, \mathbb{C}) = \sum_{i+j=k} H^{i,j}(Y, \mathbb{C})$$

and we denote

$$H^+ = H^k(Y, \mathbb{C})^+ = \sum_{i+j=k, i>j} H^{i,j}(Y, \mathbb{C})$$

$$H^- = H^k(Y, \mathbb{C})^- = \sum_{i+j=k, i<j} H^{i,j}(Y, \mathbb{C})$$

so we have

$$H^+ = \overline{H^-}; \quad H^+ \cap H^- = 0 \quad (1.1)$$

(1.1) implies that the image of  $H^k(Y, \mathbb{Z})$  in  $H^-$  under the projection along  $H^+$  will be a lattice. A complex torus  $H^k(Y, \mathbb{C})^- / \text{im } H^k(Y, \mathbb{Z})$  is called the  $k$ -th intermediate Jacobian of  $Y$ , it is denoted by  $J^k(Y)$ . For  $k = n$  it is called also the middle Jacobian.

To define a structure of a polarized abelian variety on a complex torus  $V/R$ , it is necessary to define an hermitian form  $H$  on  $V$  satisfying conditions:

- (i)  $H$  is positively defined;
- (ii)  $\text{im } H$  restricted on  $R$  takes integer values.

There exists a Hodge real bilinear form  $Q$  on  $H^k(Y, \mathbb{C})$  (see, for example, [39]) which is defined (for  $k \leq n$ ) as follows:

$$Q(\xi, \eta) = \xi \wedge \eta \wedge L^{n-k} \in H^{2n}(Y, \mathbb{C})$$

Here  $H^{2n}(Y, \mathbb{C})$  is identified with  $\mathbb{C}$ ,  $L \in H^{1,1}(Y, \mathbb{C}) \cap H^2(Y, \mathbb{Z})$  is the fundamental class of a hyperplane section of  $Y$  or, which is the same, the cohomology class that corresponds to a Kähler form on  $Y$ . Now we define  $H$  on  $H^-$  as follows:

$$H(\xi, \eta) = iQ(\xi, \bar{\eta})$$

$H$  is hermitian, because  $Q$  is skew symmetric on odd-dimensional cohomology. A map

$$\wedge L^{n-k} : H^k(Y, \mathbb{Z}) \rightarrow H^{2n-k}(Y, \mathbb{Z})$$

is an inclusion, and a pairing  $H^k(Y, \mathbb{Z}) \otimes H^{2n-k}(Y, \mathbb{Z}) \rightarrow \mathbb{Z}$  is unimodular, according a Poincaré theorem. This means that (ii) is satisfied for  $H$ , and moreover if  $\wedge L^{n-k}$  is an isomorphism then  $B(H) \stackrel{\text{def}}{=} \text{im } H$  is unimodular. Nevertheless, (i) is not satisfied in general. We denote  $k = 2l + 1$ . The Hodge - Riemann bilinear relations show that a sufficient condition for  $J^k(Y)$  to be an abelian variety is the following:  $H^- = H^{l, l+1}(Y, \mathbb{C})$  and  $H^{l-1, l}(Y, \mathbb{C}) = 0$  (i.e. all elements of  $H^{l, l+1}(Y, \mathbb{C})$  are primitive). These conditions are always true for  $k = 1$ , and  $J^1(Y) = \text{Pic}^0(Y)$ .

For  $k = n$   $\wedge L^{n-k}$  is an identical operator, hence  $B(H)$  is unimodular. Complex torus  $V/R$  having an hermitian form  $H$  on  $V$  satisfying (ii) such that  $B(H)$  is unimodular, is called a principal torus. So, a middle Jacobian  $J^n(X)$  is a principal torus (Griffiths, [38]).

For a principal torus  $T$  (like for principally polarized abelian varieties) there exists a notion of the dual torus which is isomorphic to  $T$ . For  $J^n(Y) = H^- / \text{im } H^n(Y, \mathbb{Z})$  the dual torus is  $(H^+)^* / \text{im } H_n(Y, \mathbb{Z})$ . This is a convenient way to define the Abel - Jacobi map (it is possible to define it for all  $J^k(Y)$  without consideration of the dual torus, see for example [42], but we do not need it).

Let  $F$  be a family of  $l$ -dimensional algebraic cycles on  $Y$ . The Abel - Jacobi map  $\Phi : F \rightarrow J^n(Y)$  (or  $\Phi : \text{Alb } F \rightarrow J^n(Y)$ ) is a generalization for  $n \geq 1$  of a map  $Y \rightarrow \text{Alb } Y$  for  $n = 1$ . A family of cycles  $F$  is defined by its graph  $\Gamma_F \xrightarrow{i} F \times Y$  such that for  $f \in F$   $(p_F \circ i)^{-1}(f)$  is a cycle that corresponds to  $f$ , here  $p_F$  is a projection from  $F \times Y$  to  $F$ . Then there are cylinder maps

$$\Phi_{\mathbb{Z}} : H_1(F, \mathbb{Z}) \rightarrow H_n(Y, \mathbb{Z})$$

$$\begin{array}{ccc} \Phi_{\mathbb{C}} : & H^1(F, \mathbb{C})^* & \rightarrow & H^n(Y, \mathbb{C})^* \\ & \parallel & & \parallel \\ & H_1(F, \mathbb{C}) & \rightarrow & H_n(Y, \mathbb{C}) \end{array}$$

Since  $F$  is an algebraic family,  $\Phi_{\mathbb{C}}$  behaves good with respect to the Hodge decomposition:

$$\Phi_{\mathbb{C}}(H^{i,j}(F, \mathbb{C})^*) \subset H^{i+l,j+l}(Y, \mathbb{C})^*$$

For  $(i, j) = (1, 0)$  we have  $\Phi_{\mathbb{C}}(H^{1,0}(F, \mathbb{C})^*) \subset H^{l+1,l}(Y, \mathbb{C})^*$  and hence  $\Phi_{\mathbb{C}}$  and  $\Phi_{\mathbb{Z}}$  define a map  $\Phi : \text{Alb } F \rightarrow J^n(Y)$  which is called the Abel - Jacobi map.

Further we shall consider only the middle Jacobian of threefolds. To have a possibility to distinguish rational and non-rational threefolds we should know what is the behaviour of the Jacobian under the simplest rational maps.

**Proposition 1.2.** ([32]; see also [20]). Let  $Y$  be a threefold and  $\tilde{Y}$  be the blowing up of a point (resp. of a curve  $C$ ) on  $Y$ . Then  $J^3(\tilde{Y}) = J^3(Y)$  (resp.  $J^3(\tilde{Y}) = J^3(Y) \oplus J(C)$ ).  $\square$

According Hironaka ([41]) any birational map of algebraic varieties over  $\mathbb{C}$  is a composition of blowings up and down. This implies a general theorem which is used to apply the method of middle Jacobian to proofs of non-rationality of threefolds:

**Theorem 1.3.** ([32]) Middle Jacobian of a rational threefold is an abelian variety which is isomorphic to the direct sum of Jacobians of curves.  $\square$

To apply this theorem we need to know properties of Jacobians of curves. Here is one of them:

**Theorem 1.4.** (Andreotti - Mayer criterion, [24]). Let  $(X, \Theta)$  be a principally polarized abelian variety. If  $(X, \Theta)$  is the Jacobian of a curve  $C$  then  $\text{Codim}_X \text{Sing } \Theta = 4$  (resp. 3) if  $C$  is not a hyperelliptic curve (resp. if  $C$  is a hyperelliptic curve).  $\square$

## Section 2. Fano threefolds: a brief survey.

A threefold  $X$  is called a Fano threefold if its anticanonical sheaf  $K_X^{-1}$  is ample on  $X$ . A detailed survey on Fano threefolds can be found in [2], [3], [16]. According a Kodaira criterion, a Fano threefold  $X$  has an important property  $H^i(X, O_X) = 0$  for  $i > 0$ , i.e.  $X$  satisfies an obvious necessary condition of rationality.  $J^3(X)$  is an abelian variety whose dimension is  $h^{2,1}(X)$ .

$X$  is called a Fano threefold of genus 1 if  $\text{Pic } (X) = \mathbb{Z}$ . We shall consider below only these varieties. Main invariants of  $X$  are the following: an index  $r$  is defined by the equality  $K_X^{-1} = O_X(r)$  where  $O_X(1)$  is the positive generator of  $\text{Pic } (X)$ ; a degree  $d$  is the degree of the image of  $X$  under the map defined by the linear series

$|K_X^{-1}|$ , and a genus  $g$  is the genus of the generic curve which is a section of the image of  $X$  under this map by a subspace of codimension 2.

Fano threefolds of genus one are completely classified, a table is given in [2]. All these threefolds — except  $P^3$  and a quadric threefold — have index 2 or 1. It is clear that a hyperplane section of the image of  $X$  under the map of the linear series  $|O_X(1)|$  is a Del Pezzo surface (resp. K3 surface) for  $X$  of index 2 (resp. 1). It is known that the degree of a Del Pezzo surface can be any number between 1 and 9, and the degree of a K3 surface can be any even number. But the degree of Fano threefolds of index 2 (resp. 1) can be any number between 1 and 5 (resp. any even number from 2 until 22, except 20). It is clear that for Fano threefolds of index 1 we have  $d = 2g - 2$ , and the map defined by the linear series  $|K_X^{-1}|$  sends a threefold of genus  $g$  to  $P^{g+1}$ .

Fano threefolds are counterexamples to the Luroth problem. Unirationality is proved for most types of Fano threefolds (see, for example, [16]). There exists a method of a proof of non-rationality of a Fano threefold  $X$  based on investigation of the group of birational automorphisms of  $X$ . The origin of this method is a paper of M. Noether about the Cremona group of  $P^2$ . Fano used it in [35] giving a non-strict proof of non-rationality of a quartic threefold  $X_4^3$ . A strict proof was given by Iskovskih - Manin in [6]. Later the same method was used to prove non-rationality of

- (a)  $X_2^3$  (a two-sheeted covering of  $P^3$  of index 1) ([4]);
- (b) The complete intersection of a quadric and a cubic in  $P^5$  ([5]);
- (c) A "double Veronese cone" ([22]).

The lesser is the degree of a threefold, the easier is the proof of non-rationality by this method. For example,  $X_4^3$  and  $X_2^3$  have the simplest group of birational automorphisms — all their automorphisms are regular.

At the same time Clemens and Griffiths in [32] proved non-rationality of and the global Torelli theorem for a cubic threefold by a method of intermediate Jacobian. This result stimulated analogous investigations of other Fano threefolds. Non-rationality was proved for an intersection of 3 quadrics in  $P^6$  ([21]) and the two-sheeted covering of  $P^3$  ramified at a quartic ([12]).

According [32], to prove non-rationality of a threefold  $X$  it is sufficient to show that its intermediate Jacobian  $J^3(X)$  is not the Jacobian of a curve. This can be done by investigation of Prym varieties ([27], [43]) or singularities of Poincaré divisor ([24]).

Practically the only way of investigation of  $J^3(X)$  is to study its Abel - Jacobi map  $\Phi : F \rightarrow J^3(X)$  defined for any family of curves  $F$  on  $X$ . It is a generalization of Albanese map for curves. Clemens and Griffiths in [32] considered the simplest family of curves on a cubic threefold  $X$  — its Fano surface  $F(X)$  which is the family of all straight lines on  $X$ . The main technical result that is used for a proof of non-rationality and global Torelli theorem for a cubic threefold — the description of Poincaré divisor  $\Theta(J^3(X))$  on  $J^3(X)$  — is obtained in terms of its Abel - Jacobi map.

A cubic threefold  $X$  and related varieties  $J^3(X)$ ,  $\Theta(J^3(X))$ ,  $F(X)$  were studied also in [17], [18], [23], [29], [44] — [46], in earlier Fano papers [34], [36] (this list is not exhaustive). Global Torelli theorem and the description of singularities of Poincaré divisor for a cubic threefold was proved independently in [18], [19].

Let us consider now the main steps of [32] in more details. Firstly we give some general definitions.

Let  $\phi : Y \rightarrow A$  be a map of a  $k$ -dimensional variety  $Y$  to an abelian variety  $A$ . For any  $y \in Y$  a map of tangent spaces is defined:  $T_\phi(y) : T_Y(y) \rightarrow T_A(\phi(y))$ . Let  $\phi$  satisfy a property that  $T_\phi$  is an inclusion at a generic point. Then there exists a Gauss map which is a rational map  $G(\phi) : Y \rightarrow G(k, T_A(0))$  where  $G(k, T_A(0))$  is the Grassmann variety. The Gauss map is defined as follows. For  $y \in Y$   $T_\phi(y)(T_Y(y))$  is an element of  $G(k, T_A(\phi(y)))$ . The group law on an abelian variety defines a canonical isomorphism of tangent spaces at any its points, and hence an isomorphism  $G(k, T_A(\phi(y))) \rightarrow G(k, T_A(0))$ . An image of the above element under this isomorphism is the desired element  $G(\phi)(y)$ .

Let  $\Phi : Y \rightarrow \text{Alb } Y$  be the Albanese map. If its Gauss map exists then it coincide with the map  $Y \rightarrow G(k, H^0(Y, \Omega_Y)^*)$  associated with the sheaf  $\Omega_Y$  (since  $T_{\text{Alb } Y}(0)$  is canonically isomorphic to  $H^0(Y, \Omega_Y)$ ).

Let now  $V$  be a vector space of dimension 5 and  $X \subset P(V)$  a cubic threefold. By definition, its Fano surface  $F = F(X)$  is the set of straight lines on  $X$ . It is a subvariety of  $G(2, V)$ , we denote the corresponding inclusion  $F \rightarrow G(2, V)$  by  $i_F$ .

**Theorem 2.1.**  $\dim \text{Alb } F = h^{1,0}(F) = 5$ .

There exist different ways of proof of this theorem. Altman and Kleiman ([23]) use that  $F$  is a set of zeros of a section of a locally free sheaf  $E$  of rank 4 on  $G(2, V)$ . This permits to get a resolvent

$$0 \rightarrow \lambda^4(E^*) \rightarrow \lambda^3(E^*) \rightarrow \lambda^2(E^*) \rightarrow \lambda^1(E^*) \rightarrow O_G \rightarrow O_F \rightarrow 0$$

of  $O_F$  on  $G(2, V)$  and to use it for an explicit calculation of  $H^1(F, O_F)$ . In principle, this method can be applied to all families of subvarieties on a variety, but for more complicated Fano threefolds calculations are too long.

A proof of this theorem given in [32] uses a technique of degeneration of  $X$  to a threefold  $X_0$  having one double point  $x_0$ . A set of straight lines on  $X_0$  which contain  $x_0$  is a smooth curve  $D_0$  on  $F_0 = F(X_0)$ ; it is easy to calculate that its genus is 4 and that  $\text{Sing } F_0 = D_0$ . The tangent cone to  $X_0$  at  $x_0$  has 2 rulings. This implies: if  $r : \tilde{F}_0 \rightarrow F_0$  is a desingularization of  $F_0$  then  $r^{-1}(D_0) = D_1 \cup D_2$  is a pair of curves which are isomorphic to  $D_0$ . Further, we have:  $\tilde{F}_0 = S^2(D_0)$ ,  $h^{1,0}(\tilde{F}_0) = h^{1,0}(S^2(D_0)) = g(D_0) = 4$ . Finally, degeneration theory for  $X$  and  $F$  implies that  $h^{1,0}(F) = h^{1,0}(\tilde{F}_0) + 1 = 5$ . The situation for the Fano threefolds considered in the present paper is the same (see Proposition 5.12).

Theorem 2.1 implies the following

**Corollary 2.2.** Abel - Jacobi map  $\Phi : \text{Alb } F \rightarrow J^3(X)$  is an isogeny.  $\square$

Really,  $\Phi$  is an isomorphism, but we do not need it.

The next step is a proof of the tangent bundle theorem for  $F$ . It establishes a connection between  $\Omega_F$  (or  $T_F$ ) and a sheaf defined on  $X$  "geometrically". For the case of a cubic threefold its statement is very simple:

**Theorem 2.3.** There is an isomorphism of sheaves on  $F$ :  $\Omega_F = i_F^*(\tau_{2,V}^*)$ .

A proof of this theorem for the Fano surface of a cubic threefold is also very easy (comparatively to proofs of analogous theorems for other Fano threefolds). The main idea of the proof is a calculation of  $T_F$  by means of deformation theory.

Theorems 2.1 and 2.3 imply

**Corollary 2.4.** (Geometric interpretation of the tangent bundle theorem). There exists an isomorphism  $H^0(F, \Omega_F)^* \rightarrow V$  such that the canonical map  $F \rightarrow G(2, H^0(F, \Omega_F)^*)$  coincide with the inclusion  $i_F : F \rightarrow G(2, V)$ .  $\square$

Corollaries 2.2 and 2.4 imply that the Gauss map of  $\Phi$  (i.e. a map  $G(\Phi) : F \rightarrow T_{J^3(X)}(0)$ ) coincides with  $i_F$ .

**Theorem 2.5.** (Analog of a Riemann theorem). A set of points of type  $\Phi(f_1) - \Phi(f_2)$  where  $f_1, f_2 \in F$  is (up to a shift) a Poincaré divisor  $\Theta \xrightarrow{i_\Theta} J^3(X)$ .  $\square$

Theorem 2.5 and Corollary 2.4 are used for a description of the Gauss map of  $i_\Theta$ , i.e. a map  $G(i_\Theta) : \Theta \rightarrow G(4, V) = P(V^*)$ . It is clear that if  $f_1, f_2 \in F$  and the corresponding straight lines in  $X$  do not meet one another, then (2.4) implies that  $G(i_\Theta)(\Phi(f_1) - \Phi(f_2))$  is a space  $P^3 \in P(V^*)$  spanned on  $f_1, f_2$  considered as straight lines in  $P(V)$ . It is possible to prove that if  $t$  is an intersection point of  $f_1, f_2$  considered as straight lines in  $P(V)$  then  $G(i_\Theta)(\Phi(f_1) - \Phi(f_2))$  is the tangent space of  $X$  at  $t$ .

This means that  $G(i_\Theta) : \Theta \rightarrow G(4, V) = P(V^*)$  is a  $n$ -fold covering where  $n = \#(G(i_\Theta)^*(P^3))$  for a  $P^3 \in P(V^*)$  is equal to the number of pairs of non-meeting straight lines on a smooth cubic surface  $P^3 \cap X$ . We have  $n = 216$ . This is sufficient to make a conclusion that  $J^3(X)$  is not a Jacobian of a curve  $C$  of genus 5. Really, for  $J(C)$  the degree of covering of the Gauss map  $G(i_\Theta) : \Theta \rightarrow G(4, V)$  is clearly  $\binom{8}{4} = 70$ . Another way to see that  $J^3(X) \neq J(C)$  is to investigate the ramification divisor of  $G(i_\Theta)$ . For  $J^3(X)$  (resp.  $J(C)$ ) it is the set of  $P^3$  which are tangent to  $X$  (resp. to  $C$ ). Finally, the above recovering of  $X$  by ramification divisor of  $G(i_\Theta)$  means that the global Torelli theorem for it is proved:

**Theorem 2.6.** The polarized abelian variety  $(J^3(X), \Theta)$  defines  $X$  uniquely.  $\square$

Analog of results of [32] for cubic threefolds are obtained now (1982) for the following Fano threefolds:

1. Two-sheeted covering of  $P^3$  with ramification in a quartic ([11], [12], [14], [49], [51]). Since many properties of  $X_{10}^3$  — a Fano threefold of genus 6 (the main object of the present paper) are analogous to the ones of a two-sheeted covering of  $P^3$  with ramification in a quartic, a brief survey of them will be given below.

2.  $X_8^3$  — an intersection of 3 quadrics in  $P^6$  ([21], [28], [50], [51]). Main theorems concerning  $X_8^3$  are given in Section 5, because they are necessary for a calculation of irregularity of  $X_{10}^3$ . Here there is no complete analogy: proofs of non-rationality and of global Torelli theorem for  $X_8^3$  use a fact that  $J^3(X_8^3)$  is a Prym variety. Tangent bundle theorem for  $X_8^3$  is unknown.

3. "Double cone Veronese"  $X_1^3$  ([13]). Here the situation is more complicated than for other investigated types of Fano varieties. Namely,  $F(X_1^3)$  is not reduced, and  $X_1^3$  is the only known Fano threefold for which  $h^{1,0}(F(X_1^3)) \neq h^{2,1}(X_1^3)$  (here we have  $h^{1,0}(F(X_1^3)) = 2h^{2,1}(X_1^3)$ ).

An attempt to investigate the Fano surface of a quartic threefold  $X_4^3$  was made in [33]. This subject is very difficult: for example, computer calculation was used to find a number of conics passing through a fixed generic point of  $X_4^3$  (for Fano threefolds considered above this number can be found easily by hand calculations).

Let us consider now main steps of a proof of non-rationality of a two-sheeted covering of  $P^3 \subset P(V_4)$  with ramification in a quartic  $W \subset P^3$ . Let  $v : P^3 \rightarrow P^9 =$

$P(S^2(V_4))$  be a Veronese map (i.e. defined by the complete linear series  $O_{P^3}(2)$ ),  $t$  a point in a  $P^{10} - P^9$ ,  $K$  a cone over  $v(P^3)$  and a vertex  $t$ ,  $p : K \rightarrow v(P^3)$  a projection from  $t$ ,  $\Omega$  a quadric in  $P^{10}$  such that  $t \notin \Omega$ . Then "a double  $P^3$ "  $X = K \cap \Omega$ . Without loss of generality we can suppose that  $P^9$  is the polar hyperplane for  $\Omega$  with respect to  $t$ . Then the ramification quartic  $W = v^{-1}(\Omega \cap v(P^3))$ .

Smoothness of  $X$  implies smoothness of  $W$ . We consider below only the case when  $W$  does not contain straight lines (this restriction is rejected in [15]). It is easy to see that a condition that  $W$  does not contain straight lines is analogous to a condition of the lemma 3.7 of the present paper for  $X_{10}^3$ .

If  $c$  is a conic on  $X$  then  $p(c)$  is a conic on  $v(P^3)$  and  $v^{-1}(p(c))$  is a straight line in  $P(V_4)$  which is tangential to  $W$  at 2 points (a bitangent of  $W$ ). We define a Fano surface  $F = F(X)$  of  $X$  as a set of conics on  $X$ , and we denote a set of bitangents of  $W$  in  $P(V_4)$  by  $F_0$ . A map  $c \mapsto v^{-1}(p(c))$  is a two-sheeted non-ramified covering  $p_F : F \rightarrow F_0$ , we denote the corresponding involution on  $F$  by  $i_F$ . All conics on  $X$  are smooth. Involutionary conics  $c$  and  $i_F(c)$  meet at 2 points  $\gamma_1(c), \gamma_2(c) \in v(W)$ , and  $v^{-1}(\gamma_i(c))$  are points of tangency of  $p_F(c)$  of  $W$ . There are 12 conics passing through a fixed generic point of  $X$  and 6 pairs of involutory conics passing through a fixed generic point of  $v(W)$ .

By definition,  $F_0$  is included in  $G(2, V_4) \subset P^5 = P(\lambda^2(V_4))$ . It was indicated above (in (0.11)) that (0.4) and (0.5) are true for this inclusion and hence Hodge numbers of  $F$  and  $F_0$  are given in (0.6) - (0.8).

Main theorems for a "double  $P^3$ " are the following:

**Theorem 2.7.** (Tangent bundle theorem). There exists a locally free sheaf  $\tau_2$  on  $F$  of rank 2 such that for  $c \in F$   $P(\tau_2(c))$  is a linear envelope of  $\gamma_1(c), \gamma_2(c)$ . Then  $\Omega_F = \tau_2^*$ .  $\square$

**Theorem 2.8.**  $h^{1,0}(F) = 10$ . Moreover, Abel - Jacobi map  $\Phi : \text{Alb } F \rightarrow J^3(X)$  is an isomorphism.  $\square$

**Corollary 2.9.** (Geometric interpretation of the tangent bundle theorem). Let us consider a map  $\rho_1 : F \rightarrow G(2, H^0(F, \Omega_F)^*)$  given by  $\Omega_F$ , and a map  $\rho_2 : F \rightarrow G(2, S^2(V_4))$  which sends  $c \in F$  to a linear envelope of  $\gamma_1(c), \gamma_2(c) \subset P(S^2(V_4))$ . Then there exists a natural isomorphism  $\iota : H^0(F, \Omega_F)^* \rightarrow S^2(V_4)$  such that  $\iota \circ \rho_1 = \rho_2$ .  $\square$

**Theorem 2.10.**  $F(X)$  defines  $X$  uniquely.  $\square$

**Theorem 2.11.** (Analog of a Riemann theorem). Let us consider a map  $\Phi_5 : \prod_{i=1}^5 F \rightarrow J^3(X)$  defined as follows:  $\Phi_5(f_1, \dots, f_5) = \sum_{i=1}^5 \Phi(f_i) - \Phi(i_F(f_i))$ . Then its ramification divisor is a Poincaré divisor  $\Theta$  on  $J^3(X)$ .  $\square$

**Theorem 2.12.**  $\Theta$  is irreducible, and  $\text{codim }_{J^3(X)} \text{Sing } \Theta = 2$ .  $\square$

So, according Andreotti - Meier criterion,  $J^3(X)$  is not the Jacobian of a curve, and hence  $X$  is not rational.

**Theorem 2.13.** (Global Torelli theorem).  $(J^3(X), \Theta)$  defines  $X$  uniquely.  $\square$

Let us consider now a class of threefolds for which it is easy to describe their middle jacobian, namely bundles of conics. Let  $X$  be any threefold immersed in a projective space. By definition, a structure (denoted by  $B$ ) of a bundle of conics on  $X$  is a map  $p = p(B) : X \rightarrow P^2$  such that each fibre of  $p$  is isomorphic to a

conic. Attached to  $B$  is a degeneration curve  $C = C(B) \subset P^2$  defined as follows:  $x \in C \iff p^{-1}(x)$  is a reduced conic. All singularities of  $C$  are ordinary double points, and we have:

$$x \notin \text{Sing } C \iff p^{-1}(x) \text{ is a pair of straight lines}$$

$$x \in \text{Sing } C \iff p^{-1}(x) \text{ is a double straight line}$$

Attached is also a two-sheeted covering  $p_C = p_C(B) : \hat{C} \rightarrow C$  defined as follows: for  $x \in C$   $p_C^{-1}(x)$  is a set of irreducible components of  $p^{-1}(x)$ . It has the following property: ramification points of its normalization  $p_{C_N} : \hat{C}_N \rightarrow C_N$  are exactly inverse images of singular points of  $C$  under the normalization map  $N : C_N \rightarrow C$ .

Coverings which satisfy this property are called two-sheeted pseudocoverings. Beauville ([26]) defined a notion of Prym variety for the case of pseudocoverings. There is a theorem:

**Theorem 2.14.**  $J^3(X) = \text{Pr } (\hat{C}/C)$  ([43] for a smooth  $C$ , [26] for any  $C$ ).  $\square$

There are sufficient conditions that a Prym variety is not the Jacobian of a curve, for example

**Proposition 2.15.** Let  $B$  be a bundle of conics such that  $C(B)$  is non-singular and  $\deg C(B) \geq 6$ . Then  $\text{Pr } (\hat{C}/C)$  is not the Jacobian of a curve ([43]).  $\square$

For all Fano threefolds considered above, existence of a structure of bundle of conics is known only on threefolds which are birationally equivalent to cubics and intersections of three quadrics. If  $X$  is a cubic threefold then  $\deg(C) = 5$ , and in order to prove that  $\text{Pr } (\hat{C}/C)$  is not the Jacobian of a curve we need to study the sheaf of second order  $\sigma$  on  $C$  that defines  $p_C$ . If  $X$  is an intersection of 3 quadrics then  $C$  is the Hesse curve  $H$  (see Section 5). It is proved in [12] that there is no structure of bundle of conics on "double  $P^3$ ".

For other types of Fano threefolds existence of a structure of a bundle of conics is known only for degenerated Fano threefolds. Nevertheless, there exists a method of proof of non-rationality of a "generic" Fano threefold if there exists its degeneration to a non-rational bundle of conics (see [16] for details). This method was used in [26] for a proof of non-rationality of a "generic" Fano threefold for all types of Fano threefolds of genus one whose non-rationality was not known earlier.

### Section 3. Fano surface as a determinantal variety.

Here we introduce some notations that will be used throughout the paper.

A sheaf of type  $O_\alpha(a) \otimes O_\beta(b) \otimes \dots$  will be denoted by  $O_{(\alpha a, \beta b, \dots)}$ . The inverse image of a sheaf — if the map is clear — will be denoted often by the same symbol as the sheaf itself. The blowing up map of a variety  $Y$  along  $X \subset Y$  will be denoted by  $r$  with some subscript, and for  $Z \subset Y$  the notation  $r^{-1}(Z)$  will mean the total inverse image of  $Z$ , if  $Z \subset X$ , and the proper inverse image if  $Z \not\subset X$ .  $V_\bullet$  means a vector space, and the subscript is its dimension. If  $E$  is a locally free sheaf on a variety  $Y$ , then the fibre of the corresponding vector bundle at a point  $y \in Y$  will be denoted by  $E(y)$ .  $P_Y(E) \rightarrow Y$  and  $G_Y(k, E) \rightarrow Y$  mean the geometric projectivization and grassmannization respectively, i.e.  $P_Y(E) = \text{Proj}(\sum_k S^*(E^*))$ . Members of tautological exact sequences on  $G_Y(k, E)$  are denoted as follows:

$$0 \rightarrow \tau_{k,E} \rightarrow E \rightarrow \tau_{n-k,E}^* \rightarrow 0$$

$$0 \rightarrow \tau_{n-k,E^*} \rightarrow E^* \rightarrow \tau_{k,E}^* \rightarrow 0$$

(here  $n = \dim E$ ). For  $t \in G_Y(k, V_n)$   $t_V$  means a  $k$ -dimensional subspace in  $V_n$  that corresponds to  $t$ . In some cases a vector space and its projectivization will be denoted by the same symbol.

Here we give without proofs some results about  $G(2, 5)$  and its hyperplane sections (see, for example, [48]).

We identify  $G(2, V_n)$  and its image in  $P(\lambda^2(V_n))$  under the Plücker embedding. For  $n = 4$   $G(2, 4)$  is a quadric hypersurface in  $P(\lambda^2(V_4))$ , we shall denote the corresponding element in  $P(S^2(\lambda^2(V_4^*)))$  by  $Pl(V_4)$ . Further we shall consider a space  $V = V_5$  and the Grassmannian  $G = G(2, V_5)$ .

For a conic line  $c$  we denote by  $\pi(c)$  the plane spanned on  $c$ . Let  $c$  be a conic line on  $G$ . If  $\pi(c) \not\subset G$  or if  $\pi(c) \subset G$  as an  $\alpha$ -plane (i.e. as a Schubert cycle  $\Omega(1, 4)$ ), then there exists the only  $V_4 \subset V$  such that  $c \subset G(2, V_4)$ . If  $\pi(c) \subset G$  as a  $\beta$ -plane (i.e.  $P(V_3)$  for some  $V_3 \subset V$ ), then all spaces  $V_4$  which contain  $V_3$  satisfy the condition  $c \subset G(2, V_4)$ .

Let  $H_1, H_2$  be hypersurfaces in  $\lambda^2(V)$ , i.e. elements of  $P(\lambda^2(V^*))$ , and  $E_2$  a plane that they generate in  $\lambda^2(V^*)$ . We denote  $V_8 = H_1 \cap H_2 \subset \lambda^2(V)$  and  $G_4 = G(2, V) \cap H_1 \cap H_2 \subset P(V_8)$ . We can associate to an element  $H \in \lambda^2(V^*)$  a skew-symmetric map  $S_H : V \rightarrow V^*$ , and we can associate to an inclusion  $E_2 \hookrightarrow \lambda^2(V^*)$  a map  $S : E_2 \otimes V \rightarrow V^*$  such that for  $H \in E_2$ ,  $v \in V$  we have  $S(H \otimes v) = S_H(v)$ .

For generic  $H_1, H_2$   $G_4$  is smooth. The condition of smoothness of  $G_4$  is equivalent to the condition that  $\forall H \in P(E_2)$  the rank of  $S_H$  is 4. Further, taking into consideration (3.5), we shall suppose that the pair  $H_1, H_2$  satisfies this condition.

We shall often consider the space  $P(V^*)$ , and we shall consider an element  $V_4 \in P(V^*)$  as a hyperplane in  $V$ . The tautological exact sequence on  $P(V^*)$  is the following:

$$0 \rightarrow \tau_4 \xrightarrow{i_V} V \otimes \mathcal{O} \rightarrow \mathcal{O}(1) \rightarrow 0$$

For  $V_4 \in P(V^*)$  we denote

$$M(V_4) = \lambda^2(V_4) \cap H_1 \cap H_2 \subset V_8,$$

$$Q_G(V_4) = G(2, V_4) \cap P(V_8) = G(2, V_4) \cap P(M(V_4))$$

It is clear that  $Q_G(V_4)$  is a quadric hypersurface in  $P(M(V_4))$ .

**Lemma 3.1.**  $\forall V_4 \in P(V^*) \dim M(V_4) = 4$ .

**Proof.**  $\dim \lambda^2(V_4) = 6$ . If  $\dim M(V_4) > 4$ , then there exists an element  $H \in P(E_2)$  such that  $H \supset \lambda^2(V_4)$ . This means that the composition map

$$V_4 \hookrightarrow V \xrightarrow{S_H} V^* \twoheadrightarrow V_4^*$$

is 0, hence  $S_H(V_4) \subset (V/V_4)^*$ , i.e.  $\text{rank}(S_H) = 2$  — a contradiction.  $\square$

Let us define a sheaf  $M$  on  $P(V^*)$  as the kernel of the composition

$$\lambda^2(\tau_4) \xrightarrow{\lambda^2(i_V)} \lambda^2(V) \otimes \mathcal{O} \rightarrow E_2^* \otimes \mathcal{O}$$

where the epimorphism  $\lambda^2(V) \rightarrow E_2^*$  is dual to the inclusion  $E_2 \rightarrow \lambda^2(V^*)$ . It is clear that the fibre of  $M$  at  $V_4 \in P(V^*)$  is  $M(V_4)$ . Lemma 3.1 implies that  $M$  is a locally free sheaf of rank 4.

Let a locally free sheaf  $E$  on a variety  $Y$  be given. A bundle of quadrics  $Q$  on  $(Y, E)$  is a map that associates to any  $y \in Y$  a quadric (a fibre of  $Q$  in  $y$ )  $Q(y) \subset P(E(y))$  such that they form an algebraic family. This means that there exists an invertible sheaf  $L(Q)$  on  $Y$  and a map of sheaves  $i(Q) : L(Q) \rightarrow S^2(E^*)$  such that

$$\forall y \in Y \quad i(Q)(y) = Q(y) \in P(S^2(E(y))^*)$$

So, there exists a bundle of quadrics  $Q_G$  on  $(P(V^*), M)$  whose fibre at  $V_4 \in P(V^*)$  is  $Q_G(V_4)$ .

**Lemma 3.2.** Let  $E$  be a locally free sheaf of rank 4 on  $Y$ . There exists a bundle of quadrics  $\text{Pl}(E)$  on  $(Y, \lambda^2(E))$  having  $\text{Pl}(E)(y) = \text{Pl}(E(y))$ . Then  $L(\text{Pl}(E)) = (\det E)^{-1}$ .

**Proof.** Let  $U_1, U_2 \subset Y$  be affine open subsets such that  $E|_{U_i}$  is a free  $\mathcal{O}_{U_i}$ -module. Let  $e_i = (e_{i1}, \dots, e_{i4})$  ( $i = 1, 2$ ) be its basis, and  $e_{i1}^*, \dots, e_{i4}^*$  a basis of  $E^*|_{U_i}$  which is dual to  $(e_i)$ . Further, let  $e_{ikl}^* = e_{ik}^* \wedge e_{il}^*$ . We have:

$$\text{Pl}(e_i) = e_{i12}^* \circ e_{i34}^* - e_{i13}^* \circ e_{i24}^* + e_{i14}^* \circ e_{i23}^* \in S^2(\lambda^2(E^*))|_{U_i}$$

is an equation of the Plücker quadric. Let us consider a map of sheaves  $\mathcal{O} \rightarrow S^2(\lambda^2(E^*))$  on  $U_i$  defined by the element  $\text{Pl}(e_i)$ . Let on  $U_1 \cap U_2$  we have  $e_1 = A(e_2)$  for an  $A \in GL(E|_{U_1 \cap U_2})$ . Then it is easy to check that  $\text{Pl}(e_1) = (\det A)^{-1} \text{Pl}(e_2)$  on  $U_1 \cap U_2$ . This implies the desired.  $\square$

Let us apply this lemma to  $\tau_4$  on  $P(V^*)$ . Since  $\det(\tau_4) = \mathcal{O}(-1)$ , we have: the bundle  $\text{Pl}(\tau_4)$  can be given by a map

$$i(\text{Pl}(\tau_4)) : \mathcal{O}(1) \rightarrow S^2(\lambda^2(\tau_4^*))$$

Restriction of  $\text{Pl}(\tau_4)$  on  $M \subset \lambda^2(\tau_4)$  gives us the bundle  $Q_G$ . This means that  $L(Q_G) = \mathcal{O}(1)$ .

Let us describe now the set of planes on  $G_4$ . We define an inclusion  $\psi_1 : P(E_2) \rightarrow P(V)$  as follows:  $\psi_1(H) = \text{Ker } S_H$ . Then  $\text{im } \psi_1$  is a conic line, we denote it by  $c_u$  and its plane  $\pi(c_u)$  by  $U = P(U_3)$ , where  $U_3$  is a subspace of  $V$ . We have:  $U^* = P(U_3^*) \subset G_4$  is the only  $\beta$ -plane on  $G_4$ . A straight line in  $P(V^*)$  which is dual to  $U$  will be denoted by  $l$ , i.e.  $l = P((V/U_3)^*) \subset P(V^*)$ , and a point of  $l$  is a subspace  $V_4$  such that  $U_3 \subset V_4 \subset V$ . There exists an isomorphism  $\psi : c_u \rightarrow l$  such that  $\psi(v) = S(E_2 \otimes v) \in P(V^*)$ , and we denote by  $\alpha(v)$  a Schubert cycle  $\Omega(v, \psi(v))$ . Then  $\forall v \in c_u \quad \alpha(v) \subset G_4$ , and all  $\alpha$ -planes on  $G_4$  are  $\alpha(v)$  for some  $v \in c_u$ . Further, we have  $\alpha(v) \cap U^* \subset G_4$  is a straight line, we denote it by  $l_g(v)$ . It is dual to the point  $v$  with respect to the plane  $U$ . All straight lines  $l_g(v)$  are tangent to the conic line  $c_u^* \subset U^*$  (the dual to  $c_u$ ). It is clear that  $\forall V_4 \in l$  we have  $Q_G(V_4) = U^* \cup \alpha(\psi^{-1}(V_4))$ , i.e. is a pair of planes.

**(3.3).** The inverse is also true: if for  $V_4 \in P(V^*) \quad Q_G(V_4)$  is a pair of planes, then it follows easily from the description of the set of planes on  $G_4$  that  $V_4 \in l$ .

**(3.4).** There exists a map  $b : G_4 \rightarrow P^4$  having the following properties: it is a birational isomorphism; it is the restriction to  $G_4$  of the linear projection  $p_{85}$  whose

center is  $U^*$ , from  $P(V_8)$  to  $P^4 = P(V_8/U_3^*)$ . There exist: (a) the unique normcubic  $c_3 \subset P^4$  whose linear envelope is a  $P^3 \subset P^4$ , (b) an isomorphism  $\psi_2 : c_u \rightarrow c_3$  and (c) an isomorphism  $\tilde{b} : (\tilde{G}_4)_{U^*} \rightarrow (\tilde{P}^4)_{c_3}$  - a desingularization of  $b$  such that the following diagram is commutative:

$$\begin{array}{ccc} (\tilde{G}_4)_{U^*} & \xrightarrow{\tilde{b}} & (\tilde{P}^4)_{c_3} \\ \downarrow & & \downarrow \\ G_4 & \xrightarrow{b} & P^4 \end{array}$$

where left and right vertical maps (denoted by  $r_G$  and  $r_P$  respectively) are blowings up of  $G_4$  and  $P^4$  respectively. Further, for  $v \in c_u$  we have:  $r_G^*(\alpha(v))$  is a plane, and

$$\tilde{b}(r_G^*(\alpha(v))) = r_P^*(\psi_2(v))$$

For  $t \in U^* \subset G$  we denote  $\{v_1, v_2\} = t_V \cap c_u$ . Then  $\langle \psi_2(v_1), \psi_2(v_2) \rangle$  is a bisecante of  $c_3$  in  $P^3$ , and

$$\tilde{b}(r_G^*(t)) = r_P^*(\langle \psi_2(v_1), \psi_2(v_2) \rangle)$$

For  $V_4 \in P(V^*) - l$  the map  $b|_{Q_G(V_4)} : Q_G(V_4) \rightarrow P^2$  can be described as follows.  $Q_G(V_4) \cap U^* = \{t\}$  (one point); it is clear that  $t_V = V_4 \cap U_3$ . Then  $b|_{Q_G(V_4)}$  is the blowing up of  $t$  into to a bisecante of  $c_3$  and the blowing down of  $t_1, t_2$  into intersection points of this bisecante with  $c_3$ , where  $t_1, t_2$  are straight lines on  $Q_G(V_4)$ , which contain  $t$ .

Let us describe now conic lines on  $X$ . Let  $X = G_4 \cap \Omega$ , where  $\Omega$  is a quadric hypersurface on  $P(V_8)$ .

**Lemma 3.5.**  $G_4$  is smooth.

**Proof.** If  $G_4$  is not smooth, then  $\exists H \in P(E_2)$  such that  $\text{rank } S_H = 2$ . Then  $\text{Sing } (G \cap H) = G(2, \text{Ker } S_H) = P^2$ , hence  $\text{Sing } X \supset P^2 \cap H_1 \cap \Omega$ . This means that  $\text{Sing } X$  is not empty - a contradiction.  $\square$

We denote by  $F_c = F_c(X)$  the set of conics on  $X$ ; it is an algebraic variety which is called the Fano surface of  $X$  (later we shall see that  $F_c$  is really a surface). Taking into consideration that for a generic conic  $c \subset G \exists! V_4 \in P(V^*)$  such that  $c \subset G(2, V_4)$ , we define a surface  $F = F(X)$  (which we shall call the Fano surface as well) as a set of pairs  $(c \in F_c, V_4 \in P(V^*))$  such that  $c \subset G(2, V_4)$ . There are natural projections

$$r_F : F \rightarrow F_c, \quad \tilde{\phi} : F \rightarrow P(V^*)$$

There exists a bundle of quadrics  $Q_\Omega$  on  $(P(V^*), M)$  whose fibre at  $V_4 \in P(V^*)$  is  $Q_\Omega(V_4) = P(M(V_4)) \cap \Omega$ . It is clear that

$$(a) \ L(Q_\Omega) = O;$$

$$(b) \ X \cap G(2, V_4) = Q_G(V_4) \cap Q_\Omega(V_4).$$

Further, we have  $\forall V_4 \in P(V^*) \quad \dim Q_G(V_4) \cap Q_\Omega(V_4) = 1$ , because smoothness of  $X$  implies that  $\text{Pic } (X)$  is generated by  $O_{P(V_8)}(1)|_X$ , so the degree of any surface on  $X$  is a multiple of 10 ( $= \deg X$ ). But if  $Q_G(V_4) \cap Q_\Omega(V_4)$  contains a surface, then its degree is 1 or 2.

It is clear that  $(c, V_4) \in F \iff c \subset Q_G(V_4) \cap Q_\Omega(V_4)$ . Since  $Q_G(V_4) \cap Q_\Omega(V_4)$  is a curve of degree (2,2) on a quadric surface, and  $c$  is a curve of degree (1,1), then  $Q_G(V_4) \cap Q_\Omega(V_4) - c$  is also a conic line. We denote it by  $c'$ . So, there exists an involution  $i_F$  on  $F$  defined as follows:  $i_F(c, V_4) = (c', V_4)$ . Let  $F_0$  be the quotient surface  $F/i_F$  and  $p_F : F \rightarrow F_0$  the corresponding two-sheeted covering. A point  $f_0 \in F_0$  will be denoted as follows:  $f_0 = (c, c', V_4)$ , where  $f_0 = p_F(c, V_4) = p_F(c', V_4)$ . It is clear that  $c$  and  $c'$  have 2 (possible coinciding) intersection points which we denote by  $\gamma_1(f_0)$  and  $\gamma_2(f_0)$  (or  $\gamma_1(c)$  and  $\gamma_2(c)$ ). Their linear envelope (i.e. the straight line  $\langle \gamma_1(c), \gamma_2(c) \rangle$ ) will be denoted by  $l_8(c)$  (or  $l_8(f_0)$ ).

**(3.6).** There exist smooth threefolds  $X$  which contain a pair of involutory conics  $c$  and  $c'$  such that both of them is a pair of straight lines such that  $l_8(c)$  is their common component. Considerations analogous to ones used in the proof of Lemma 3.7 show that the set of  $X$  having this property has codimension 1 in the set of all possible  $X$ . These  $X$  will not be considered in the present paper, although most likely main results which we have got for a "generic"  $X$ , are true for them as well (for example, smoothness of  $F$ ).

It is clear that there exists a map  $\phi : F_0 \rightarrow P(V^*)$  satisfying a condition  $\phi \circ p_F = \tilde{\phi}$ . Further, it is clear that

$$(c, V_4) \in F \iff \pi(c) \cup \pi(c') \in \langle Q_G(V_4), Q_\Omega(V_4) \rangle_{P(S^2(M(V_4)))^*}$$

i.e. some linear combination of  $Q_G(V_4)$ ,  $Q_\Omega(V_4)$  is a pair of planes. So, let us define  $Z$  as a variety of pairs  $(V_4, Q)$  where  $V_4 \in P(V^*)$ ,  $Q \in \langle Q_G(V_4), Q_\Omega(V_4) \rangle$  together with a natural projection  $\eta : Z \rightarrow P(V^*)$ . Since  $L(Q_G) = O(1)$ ,  $L(Q_\Omega) = O$ , we have

$$Z = P_{P(V^*)}(O + O(1))$$

There is a tautological exact sequence on  $Z$ :

$$0 \rightarrow O_\eta(-1) \xrightarrow{i_\eta} O + O(1) \rightarrow O(1, \eta 1) \rightarrow 0$$

as well as the dual sequence. Further,  $Z$  is the blowing up of  $P^5 = P(V_6)$  at a point  $t \in P^5$ ; we denote the corresponding map by  $r_Z : Z \rightarrow P^5$ . It is clear that  $V_6/t = V^*$  and  $r_Z^*(O_{P^5}(1)) = O(1, \eta 1)$ . There are divisors  $D_G, D_\Omega$  on  $Z$  which are defined as follows: a point of  $Z$  can be considered as a pair  $(V_4, Q)$ ; we have:

$$(V_4, Q) \in D_G \iff Q = Q_G(V_4);$$

$$(V_4, Q) \in D_\Omega \iff Q = Q_\Omega(V_4)$$

It is clear that  $D_G = r_Z^*(t)$  is the exceptional divisor on  $Z$ , and  $r_Z(D_\Omega)$  is a hypersurface in  $P^5$ .

Let  $(V_4, Q) \in Z$ ; we associate it the quadric  $Q \subset P(M(V_4))$ . This gives us a bundle of quadrics (denoted by  $T$ ) on  $(Z, M)$ . It is clear that  $L(T) = O_\eta(-1)$  and

$$i(T) = (i(Q_G) \oplus i(Q_\Omega)) \circ i_\eta : O_\eta(-1) \rightarrow S^2(M^*)$$

There exists an inclusion  $\bar{\phi} : F_0 \rightarrow Z$  defined as follows:  $\bar{\phi}(c, c', V_4) = (V_4, \pi(c) \cup \pi(c'))$ . It is clear that  $z \in \text{im } \bar{\phi} \iff T(z)$  is a pair of planes, i.e. a rank 2 quadric surface. This means that  $F_0 \hookrightarrow Z$  is the second determinantal of  $T$ .

Let us recall the definition and main properties of determinantal varieties. Let  $E$  be a locally free sheaf of rank  $n$  on an algebraic variety  $Y$ ,  $Q$  a bundle of quadrics on  $(Y, E)$  and  $i(Q) : L^* \rightarrow S^2(E^*)$  the corresponding map. For any  $y \in Y$  we can consider the fibre  $i(Q)(y)$  as a symmetric map from  $E(y)$  to  $E(y)^*$ . The determinantal  $D_k = D_k(Q)$  of  $Q$  is defined as a set of points  $y \in Y$  such that  $\dim \operatorname{im} i(Q)(y) \leq k$ . We define also a variety  $\tilde{D}_k = \tilde{D}_k(Q)$  as a set of pairs  $(y, V_k)$  such that  $y \in D_k$  and  $\operatorname{im} i(Q)(y) \subset V_k \subset E(y)^*$ . The natural projection  $\pi_k : \tilde{D}_k \rightarrow D_k$  is an isomorphism outside of  $\pi_k^{-1}(D_{k-1})$ . There is also a tautological inclusion  $\tilde{D}_k \rightarrow G_Y(k, E^*)$ , and we have a commutative diagram:

$$\begin{array}{ccc} \tilde{D}_k & \hookrightarrow & G_Y(k, E^*) \\ \downarrow & & \downarrow \\ D_k & \hookrightarrow & Y \end{array}$$

There is a sheaf  $C = C(k) = \operatorname{Coker} (S^2(\tau_{k,E^*}) \rightarrow S^2(E^*))$  on  $G_Y(k, E^*)$ . The sheaf  $L \otimes C$  has a distinguished section coming from the map

$$O \xrightarrow{i(Q) \otimes L} L \otimes S^2(E^*) \rightarrow L \otimes C$$

It is clear that  $\tilde{D}_k$  is the set of zeros of this section.  $Q$  is called  $k$ -regular, if for  $i = k$  and  $i = k - 1$

$$\operatorname{Codim}_{G_Y(i, E^*)} \tilde{D}_i = \dim L \otimes C(i)$$

(if  $D_i \neq \emptyset$ ). If so, we have  $\operatorname{Codim}_Y D_k = \frac{(n-k)(n-k+1)}{2}$ . Further, for a  $k$ -regular bundle  $Q$  we have:  $\bar{D}_{k-1} = \pi_k^{-1}(D_{k-1})$  is a divisor in  $\tilde{D}_k$ , which is the determinantal of rank  $k - 1$  of a bundle of quadrics  $L^* \rightarrow S^2(\tau_k)$ , which is defined naturally on  $\tilde{D}_k$ . Porteus formula ([7]) implies that

$$O_{\tilde{D}_k}(\bar{D}_{k-1}) = L^k \otimes O_G(-2)$$

Particularly, if  $D_{k-1} = \emptyset$ , then  $\bar{D}_{k-1} = \emptyset$  as well, and  $L^k = O_G(2)$  in  $\operatorname{Pic}(D_k)$ .

Let us consider the case of even  $k$ . For a quadric  $Q$  of rank  $k$  the set of projective spaces of the maximal possible dimension which are contained in  $Q$ , has 2 connected components. We denote this set of connected components by  $\Gamma(Q)$ , and we define a two-sheeted covering  $p_{D_k} : \hat{\tilde{D}}_k \rightarrow \tilde{D}_k$  as follows:  $p_{D_k}^{-1}(y) = \Gamma(Q(y))$  for any  $y \in \tilde{D}_k - \bar{D}_{k-1}$ . We have:  $\bar{D}_{k-1}$  is the ramification divisor of  $p_{D_k}$ , and a sheaf that corresponds to  $p_{D_k}$  is  $L^{\frac{k}{2}} \otimes O_G(-1)$ .

Let us consider the case  $k = 2$ . For  $y \in D_2$   $Q(y)$  is a pair of hyperplanes in  $P(E(y))$ , and a choice of a point in  $p_{D_2}^{-1}(y)$  is a choice of one of these hyperplanes. So, there exists an inclusion  $\hat{\tilde{D}}_2 \rightarrow P_Y(E)$ , and there exist sheaves  $\tau_{n-1,E}$  and  $O_\pi(1)$  on  $\hat{\tilde{D}}_2$  (here  $\pi : P_Y(E) \rightarrow Y$  is a projection). It is clear that  $\hat{\tilde{D}}_2$  is the set of zeros of a section of  $L \otimes S^2(\tau_{n-1,E}^*)$  (here  $L \otimes S^2(\tau_{n-1,E}^*)$  is a sheaf on  $P_Y(E^*)$ ).

Let us return to study of the Fano surface. As we have seen,  $F_0 = D_2(T)$ .

**Lemma 3.7.** For a generic  $X$  we have  $D_1(T) = \emptyset$ .

**Proof.** It is clear that

$$D_1(T) \neq \emptyset \iff \exists V_4 \in P(V^*), \exists V_3 \subset M(V_4)$$

such that the double plane  $2P(V_3) \in \langle Q_G(V_4), Q_\Omega(V_4) \rangle$ . Let us fix  $G_4$ . For given  $V_4 \in P(V^*)$ ,  $V_3 \subset M(V_4)$   $Q_G(V_4)$  is defined uniquely, and the set of quadrics  $Q_\Omega(V_4)$  such that  $2P(V_3) \in \langle Q_G(V_4), Q_\Omega(V_4) \rangle$  has codimension 8 in the space of all quadrics in  $P(V_8)$ . This means that the set of quadrics  $\Omega$  such that  $2P(V_3) \in \langle Q_G(V_4), Q_\Omega(V_4) \rangle$  has codimension 8 in the space of all quadrics in  $P(V_8)$ . But the set of pairs  $\{V_4 \in P(V^*), V_3 \subset M(V_4)\}$  is 7-dimensional, hence the set of quadrics  $\Omega$  such that for  $X = G_4 \cap \Omega$   $D_1(T) \neq \emptyset$  is a union of a 7-dimensional set of subvarieties of codimension 8, hence has codimension  $\geq 1$ .  $\square$

(4.2) (see below) implies that  $\dim F = 2$ . This means that a condition  $D_1(T) = \emptyset$  for  $X$  is equivalent to a condition of 2-regularity of the bundle  $T$ . Further we shall consider only  $X$  that satisfy this condition.

Properties of determinantal varieties given above imply the following

**Proposition 3.8.** There exist inclusions

$$F_0 \rightarrow G_Z(2, M^*), \quad F \rightarrow P_Z(M^*)$$

hence tautological sheaves on  $G_Z(2, M^*)$  and  $P_Z(M^*)$  can be restricted on  $F_0$  and  $F$  respectively. Further,  $F_0$  is the set of zeros of a section of  $O_\eta(1) \otimes C(2)$  on  $G_Z(2, M^*)$ , and  $F$  is the set of zeros of a section of  $O_\eta(1) \otimes S^2(\tau_{3,M}^*)$  on  $P_Z(M^*)$ . We have equalities of sheaves  $O_G(2) = O_\eta(2)$  (respectively  $O_G(1) = O_\eta(1)$ ) on  $F_0$  (respectively on  $F$ ). The covering  $p_F : F \rightarrow F_0$  is non-ramified, and the corresponding sheaf is  $\sigma = O(\eta 1, G - 1)$ . For  $f_0 \in F_0$   $l_8(f_0) = P(\tau_{2,M}(f_0))$ .  $\square$

If for  $V_4 \in P(V^*)$   $Q_G(V_4)$  or  $Q_\Omega(V_4)$  is already a pair of planes then it is clear that  $V_4 \in \phi(F_0)$ . We denote the set of these  $V_4$  (and the set of their  $\phi$ -inverse images on  $F_0$ ) by  $l_G$  and  $l_\Omega$  respectively. They are curves on  $F_0$ , and it is clear that  $l_G = \bar{\phi}^*(D_G)$ ,  $l_\Omega = \bar{\phi}^*(D_\Omega)$ .  $l_\Omega$  is not defined by  $X$  uniquely, it depends on a choice of  $\Omega$  such that  $X = G_4 \cap \Omega$ . (3.3) implies that  $l_G = l$ . It is clear that  $p_F^{-1}(l)$  is a pair of straight lines, we denote them by  $l_1$  and  $l_2$  by such a way that

$$(c, V_4) \in l_1 \iff \{c = U^* \cap \Omega, V_4 \in l\}$$

$$(c, V_4) \in l_2 \iff \{c = \alpha(\psi^{-1}(V_4)) \cap \Omega, V_4 \in l\}$$

It is clear that  $p_F|_{l_i} : l_i \rightarrow l$  are isomorphisms ( $i = 1, 2$ ) and that  $r_F(l_1)$  is a point  $c_\Omega \in F_c$ ; the corresponding conic line  $U^* \cap \Omega$  on  $X$  is also  $c_\Omega$ . Using properties of conic lines on  $G$  we see that for  $c \in F_c$   $r_F^{-1}(c)$  is either a point or a straight line on  $F$ . If  $r_F^{-1}(c)$  is a straight line, then  $\pi(c)$  is a  $\beta$ -plane on  $G$ . In this case  $\pi(c) \cap H_1 \cap H_2 \supset c$ , hence  $\pi(c) \subset G_4$  and  $\pi(c) = U^*$ ,  $c = c_\Omega$ . So,  $c_\Omega$  is the only point of  $F_c$  such that  $r_F^{-1}(c_\Omega)$  is not a point. This means that  $r_F : F \rightarrow F_c$  is the blowing up of  $c_\Omega$ .

**Proposition 3.9.**  $O_{F_0}(l) = O_\eta(1)$ ,  $O_{F_0}(l_\Omega) = O(1, \eta 1)$ .

**Proof.** This important proposition (it implies smoothness of  $F_c$  in  $c_\Omega$ , see below) can be proved by different ways. Here is one of them. Let us consider maps of sheaves on  $G_Z(2, M^*)$ :

$$0 \rightarrow O_\eta(-1) \xrightarrow{i_\eta} O \oplus O(1) \xrightarrow{i_1} O(1, \eta 1) \rightarrow 0 \quad (3.10)$$

$$O(1) \xrightarrow{i_2} O \oplus O(1) \xrightarrow{i_3} S^2(M^*) \xrightarrow{i_4} C(2) \quad (3.11)$$

where  $i_3 = i(Q_\Omega) \oplus i(Q_G)$ . By definition, for  $f_0 \in F_0$   $i_4 \circ i_3 \circ i_\eta(f_0) = 0$ , and exactness of (3.10) implies that there exists an inclusion  $i_5 : O(1, \eta 1) \rightarrow C(2)$  on  $F_0$  satisfying a condition  $i_5 \circ i_1 = i_4 \circ i_3$ . It is clear that

$$f_0 \in l \iff \text{im } (i_3 \circ i_2(f_0)) = \text{im } (i_3 \circ i_\eta(f_0))$$

This implies that for  $f_0 \in F_0$   $f_0 \in l \iff i_1 \circ i_2(f_0) = 0$ , i.e.  $l$  is the set of zeros of  $i_1 \circ i_2$  on  $F_0$ . This means that  $O_{F_0}(l) = O_\eta(1)$ . Replacing in (3.11)  $O(1)$  by  $O$ , we get analogously that  $O_{F_0}(l_\Omega) = O(1, \eta 1)$ .  $\square$

We denote the projection  $P_Z(M^*) \rightarrow Z$  by  $\pi$ , and its inversible sheaf by  $O_\pi(1)$ . Further, we denote  $c_1(O(1)) = H$ ,  $c_1(O_\eta(1)) = E$ ,  $c_1(O_\pi(1)) = P$ . Chow ring  $A(P_Z(M^*))$  is generated by  $H, E, P$  satisfying relations

$$H^5 = 0, E^2 = -EH, P^4 = -3P^3H - 5P^2H^2 - 5PH^3.$$

**Proposition 3.13.** Intersection indices on  $F_0$  are:  $EH = 1, E^2 = -1$ .

**Proof.** Equality  $EH = 1$  is obvious, because  $\text{cl } (l) = E$ ,  $l$  is a straight line in  $P(V^*)$ , and  $H$  is a class of a hyperplane section in  $P(V^*)$ . Equality  $E^2 = -EH$  is true even on  $Z$  and moreover on  $F_0$ .  $\square$

**Corollary 3.14.**  $c_\Omega$  is a non-singular point on  $F_c$ .

**Proof.**  $\langle E^2 \rangle_F = -2$ .  $i_F$  interchanges  $l_1$  and  $l_2$  and hence  $\langle l_1^2 \rangle_F = \langle l_2^2 \rangle_F$ . Further,  $l_1 \cap l_2 = \emptyset$ , hence  $\langle l_1^2 \rangle_F = \langle l_2^2 \rangle_F = -1$ . This implies that  $c_\Omega$  is a non-singular point on  $F_c$ .  $\square$

Maps of blowing down of  $l_1 \cup l_2$  on  $F$  and of  $l$  on  $F_0$  are denoted by  $r : F \rightarrow F_m$  and  $r_0 : F_0 \rightarrow F_{0m}$  respectively. There exists an involution  $i_{F_m} : F_m \rightarrow F_m$  and a projection  $p_{F_m} : F_m \rightarrow F_{0m}$  making the following diagram commutative:

$$\begin{array}{ccccc}
 & F & & F & \\
 & \swarrow r_F & \searrow p_F & \xrightarrow{i_F} & \searrow p_F \\
 F_c & & & & \\
 & \searrow & \downarrow & & \downarrow \\
 & & F_0 & & \\
 & & \downarrow i_{F_m} & & \\
 & & F_m & & F_m \\
 & & \searrow p_{F_m} & \xrightarrow{i_{F_m}} & \searrow p_{F_m} \\
 & & & & F_{0m}
 \end{array}$$

where vertical maps are  $r, r_0, r$  respectively.

Let us consider a map  $r_Z \circ \bar{\phi} : F_0 \rightarrow P^5$ . Since  $l \subset D_\Omega = r_Z^*(t)$ , we have:  $r_Z \circ \bar{\phi}$  factors through  $\bar{\phi}_m : F_{0m} \rightarrow P^5$ , hence we have a commutative diagram

$$\begin{array}{ccc}
 F_0 & \xrightarrow{\bar{\phi}} & Z \\
 \downarrow & & \downarrow \\
 F_{0m} & \xrightarrow{\bar{\phi}_m} & P^5
 \end{array}$$

where vertical maps are  $r_0$  and  $r_Z$  respectively. Further, we have  $l = F_0 \times_Z D_G$ . This implies

**Proposition 3.15.**  $\bar{\phi}_m$  is regular at  $c_{0\Omega}$ , and  $\bar{\phi}_m(c_{0\Omega}) = t$ .  $\square$

Since  $l_\Omega = \bar{\phi}^*(D_\Omega)$  and  $r_Z(D_\Omega)$  is a hyperplane in  $P^5$ , we have that  $\bar{\phi}_m(l_\Omega)$  is a hyperplane section of  $\bar{\phi}_m(F_{0m})$ . We denote by  $P^{5'}$  the set of quadrics in  $P(V_8)$  that contain  $X$ ; it is the linear envelope of  $\Omega$  and  $q(P(V))$  (the definition of  $q(P(V))$  is given in the beginning of Section 5). To each  $\Omega' \in P^{5'} - q(P(V))$  we can associate its own straight line  $l_{\Omega'}$  and hyperplane  $r_Z(D_{\Omega'})$  in  $P^5$ . It is easy to see that the map  $\Omega' \rightarrow r_Z(D_{\Omega'})$  gives us a natural isomorphism  $P^{5'} \rightarrow (P^5)^*$  such that quadrics in  $q(P(V))$  correspond to hyperplanes that contain  $t$ .

**Lemma 3.16.**  $\deg \phi(l_\Omega) = 40$ .

**Proof.** We have  $l_\Omega = D_Z(Q_\Omega)$  on  $P(V^*)$ . According the Porteus formula ([7]), we have:  $\deg \phi(l_\Omega) = 4(c_1(M^*)c_2(M^*) - c_3(M^*))$ . Further,  $M = 8O - 5O(1) + O(2)$  in  $K_0(P(V^*))$ , hence  $c_t(M^*) = 1 + 3H + 5H^2 + 5H^3$ , and  $\deg \phi(l_\Omega) = 40$ .  $\square$

Since  $\text{cl}_{F_0}(l_\Omega) = H + E$ , we have  $\deg l_\Omega = \langle H + E, H \rangle_{F_0}$ , and  $\langle H^2 \rangle_{F_0} = \deg \phi(F_0) = 39$ . It is easy to see that  $\deg_{P^5}(\bar{\phi}_m(F_{0m})) = 40$  and  $\text{cl}_Z(\bar{\phi}(F_0)) = 40H^3 + 39H^2E$  (this formula can be got by application of the Porteus formula for  $T$ , but calculations will be longer than in Lemma 3.16).

**Remark.**  $\langle H^2 \rangle_{F_0}$  is equal to the number of conics passing through a fixed generic point on  $X$ . Really, let  $x \in X$ ,  $x_V$  the corresponding straight line in  $P(V)$  and  $\tilde{x}_V = P(V/x_V)^* \subset P(V^*)$  the dual plane in  $P(V^*)$ , i.e. the set of  $P^3$  that contain  $x_V$ . Let  $f = (c, V_4) \in F$  and  $x \in c$ . Then  $x_V \subset P(V_4)$ , i.e.  $V_4 \in \tilde{x}_V$ . But  $V_4 \in \phi(F_0)$ , i.e. if  $x \in c$ , then  $\tilde{\phi}(f) \in \tilde{x}_V \cap \phi(F_0)$ . Clearly the inverse is also true, so the number of conics passing through  $x$  is equal to the number of intersection points of  $\phi(F_0)$  and  $\tilde{x}_V$ . So, there are 39 conics passing through a generic point of  $X$ .

We have an equality in  $K_0(P_Z(M^*))$ :

$$\Omega_F = \Omega_{P_Z(M^*)} - O_\eta(-1) \otimes S^2(\tau_{3,M}) \quad (3.17)$$

We deduce easily from this formula that  $\omega_F = O(3, \eta 4)$ . Further, we have an exact sequence (where  $i$  is an inclusion  $l_1 \cup l_2 \rightarrow F$ )

$$0 \rightarrow r^*(\Omega_{F_m}) \rightarrow \Omega_F \rightarrow i_*(\Omega_{l_1 \cup l_2}) \rightarrow 0 \quad (3.18)$$

It implies that  $r^*(\omega_{F_m}) = O(3, \eta 3)$ , i.e.  $\omega_{F_m} = p_{F_m}^*(\bar{\phi}_m^*(O_{P^5}(3)))$ . Let  $\sigma_m$  be a sheaf of order 2 that corresponds to a two-sheeted covering  $p_{F_m}$ ; it is clear that  $r_0^*(\sigma_m) = \sigma = O_{(G-1, \eta 1)}$ . This means that  $\omega_{F_0} = O(3, G 1, \eta 3)$  and  $\omega_{F_{0m}} = \bar{\phi}_m^*(O_{P^5}(3)) \otimes \sigma_m$ . Since  $\deg \bar{\phi}_m(F_{0m}) = 40$ , we get that  $c_1(\Omega_{F_{0m}})^2 = 360$  and  $c_1(\Omega_{F_m})^2 = 720$ .

To calculate  $c_2(\Omega_F)$ , it is necessary to find  $\text{cl}(F)$  in  $A(P_Z(M^*))$ , which is equal  $c_6(O_\eta(1) \otimes S^2(\tau_{3,M}^*))$  (see (3.8)). A calculation gives the desired result:

$$\text{cl}(F) = 80P^3H^3 + 240P^2H^4 + 78P^3H^2E + 235P^2H^3E$$

Further, we get (taking into consideration (3.12)) intersection indices of divisors on  $F$ :

$$H^2 = 78; P^2 = -13; PH = 1; EH = 2; EP = -1$$

We deduce from (3.17) after some calculations:

$$c_2(\Omega_F) = 4H^2 - 4P^2 + 13EH + 4EP = 386$$

and using (3.18) we get

$$c_2(\Omega_{F_m}) = 384$$

We define a surface  $W \subset X$  as a set of intersection points of involutory conics, i.e.

$$W = \bigcup_{f_0 \in F_0} (\gamma_1(f_0) \cup \gamma_2(f_0))$$

Further, we define a surface  $\bar{W}$  which makes the following diagram commutative:

$$\begin{array}{ccc} \bar{W} & \hookrightarrow & P_{F_0}(\tau_{2,M}) \\ & \searrow p_W & \downarrow \\ & & F_0 \end{array}$$

where the vertical map is  $p$ , and  $p_W$  is a two-sheeted covering of  $F_0$  such that for  $f_0 \in F_0$  we have

$$p_W^{-1}(f_0) = \{\gamma_1(f_0), \gamma_2(f_0)\} \subset p^{-1}(f_0) = l_8(f_0)$$

It is clear that there exists a map  $\xi : P_{F_0}(\tau_{2,M}) \rightarrow P(V_8)$ , that corresponds to an inclusion of sheaves  $\tau_{2,M} \hookrightarrow M \hookrightarrow V_8 \otimes \mathcal{O}$  on  $F_0$ . We have:  $\xi|_W : \bar{W} \rightarrow W$  is an epimorphism and  $\xi^*(\mathcal{O}_{P(V_8)}(1)) = \mathcal{O}_P(1)$ .

**Lemma 3.19.** Let  $f_a = (c_a, c'_a, V_{4a}) \in F_0 - l$ ,  $f_b = (c_b, c'_b, V_{4b}) \in F_0$  and  $\gamma_1(f_a) = \gamma_1(f_b)$ . Then  $j = j(f_a, f_b) = P(M(V_{4a}) \cap M(V_{4b}))$  is a straight line in  $G_4$  which is tangent to  $\Omega$ .

**Proof.** Since  $P(V_{4a} \cap V_{4b})$  is a plane, we have  $P(\lambda^2(V_{4a} \cap V_{4b})) \subset G$  and  $j = P(\lambda^2(V_{4a} \cap V_{4b})) \cap H_1 \cap H_2$  is a linear subspace in  $G_4$ . Since  $f_a \notin l$ ,  $j$  is not a plane. If  $j$  is a point, then  $j = \gamma_1(f_a) = \gamma_1(f_b)$  and  $T_{P(M(V_{4a}))}(j)$ ,  $T_{P(M(V_{4b}))}(j)$  are linearly independent in  $T_{P(V_8)}(j)$ . But  $\dim T_X(j) \cap T_{P(M(V_{4a}))}(j) = 2$ , because this space coincide with  $T_{Q_G(V_{4a})}(j)$  or  $T_{Q_\Omega(V_{4a})}(j)$ . Analogously,  $\dim T_X(j) \cap T_{P(M(V_{4b}))}(j) = 2$ , and linear independence of these spaces implies that  $\dim T_X(j) \geq 4$  that contradicts to non-singularity. This means that  $j$  is a straight line. It is easy to see that it is tangential to  $\Omega$  at  $\gamma_1(f_a)$ .  $\square$

**Lemma 3.20.** Let  $\gamma$  be a straight line on  $G_4$  which is tangential to  $\Omega$  at a point  $t_0$ . Let  $\pi(j) = \cup_{t \in j} t_V$  be a plane in  $P(V)$  and  $\bar{\pi}(j)$  the dual straight line in  $P(V^*)$ , i.e. the set of subspaces  $P(V_4) \in P(V^*)$  that contain  $\pi(j)$ . If  $(c, c', V_4) \in F_0$  and  $V_4 \in \bar{\pi}(j)$ , then  $t_0 \in \{\gamma_1(c), \gamma_2(c)\}$  or  $j \subset X$ .

**Proof.** The above conditions imply that  $j \subset Q_G(V_4)$ . It is clear that  $j \cap \Omega = \{j \cap c, j \cap c'\}$ . Since  $j$  is tangential to  $\Omega$  at a point  $t_0$ , then:

If  $j \not\subset X$ , then  $t_0 = j \cap c = j \cap c'$ ,

and hence  $t_0 \in c \cap c'$ .  $\square$

**Proposition 3.21.** For a generic point  $t \in W$   $(\xi|_{\bar{W}})^{-1}(t)$  is one point.

**Proof.** Let for a point  $t \in W$   $(\xi|_{\bar{W}})^{-1}(t)$  is more than one point.  $f_a, f_b \in p_W(\xi|_{\bar{W}})^{-1}(t)$  and  $j = j(f_a, f_b)$ . Then a straight line  $\widetilde{\pi(j)}$  meets  $\phi(F_0)$  at points  $\phi(f_a), \phi(f_b)$ . But a variety  $B \subset G(2, V^*)$  of bisecants of  $\phi(F_0)$  is 4-dimensional, while a variety  $B'$  of straight lines on  $G_4$  which are tangential to  $\Omega$  is 3-dimensional. Let  $j$  be such a line. We can associate it a straight line  $\widetilde{\pi(j)}$  in  $P(V^*)$ . These lines form a threefold  $B' \subset G(2, V^*)$ . Intersection of  $B$  and  $B'$  in  $G(2, V^*)$  is a curve, hence the set of points  $t \in W$  such that  $(\xi|_{\bar{W}})^{-1}(t)$  is more than one point is 1-dimensional.  $\square$

**Proposition 3.22.**  $O_X(W) = O_X(21)$ .

**Proof.**  $\forall f_0 \in F_0$   $\{\gamma_1(f_0), \gamma_2(f_0)\}$  is a quadric in  $P(\tau_{2,M}(f_0))$ , and these quadrics form a bundle  $\Gamma$  on  $(F_0, \tau_{2,M})$ . Let us find  $L(\Gamma)$ . There are maps of sheaves on  $F_0$ :

$$O_\eta(-1) \xrightarrow{i_\eta} O \oplus O(1) \rightarrow S^2(M^*) \rightarrow S^2(\tau_{2,M}^*)$$

Their composition is 0, hence there exists a map

$$O(1, \eta 1) = \text{Coker } i_\eta \rightarrow S^2(M^*) \rightarrow S^2(\tau_{2,M}^*)$$

It is easy to see that this map is  $i(\Gamma)$  and  $L(\Gamma) = O(1, \eta 1)$ . This implies that

$$O_{P(\tau_{2,M})}(\bar{W}) = O_{(P^2, -1, \eta -1)} \quad (3.23)$$

We denote  $c_1(O_{P(V_8)}(1))$  by  $R$ . We have

$$\deg W = \langle W, R, R \rangle_{P(V_8)} = \langle \bar{W}, R, R \rangle_{P(\tau_{2,M})}$$

In  $A(P_{F_0}(\tau_{2,M}))$  we have an equality  $R^2 = -c_1 R - c_2$ , where  $c_1 = c_1(\tau_{2,M}) = -3H - E$ ,  $c_2 = c_2(\tau_{2,M})$ . (3.23) implies that  $\text{cl } \bar{W} = 2R - H - E$ . This means that

$$\begin{aligned} \langle \bar{W}, R, R \rangle_{P(\tau_{2,M})} &= (2R^3 - HR^2 - ER^2)_{P(\tau_{2,M})} \\ &= ((2c_1^2 - 2c_2 + Hc_1 + Ec_1)R + 2c_1c_2 + Hc_2 + Ec_2)_{P(\tau_{2,M})} \\ &= (2c_1^2 - 2c_2 + Hc_1 + Ec_1)_{F_0} = 210. \end{aligned}$$

Since  $\deg X = 10$ , we have  $O_X(W) = O_X(21)$ .  $\square$

Let us compare properties of a threefold  $X$  under consideration and another threefold  $X'$  which is a two-sheeted covering of  $P^3$  with ramification at a quartic surface  $W$  (see [11], [12], [49]).  $c_1(\Omega)^2$  and  $c_2(\Omega)$  of their Fano surfaces  $F(X')$  and  $F_m(X)$  coincide. Since irregularities coincide as well:

$$h^{1,0}(F_m(X)) = h^{1,0}(F(X')) = 10, \quad h^{1,0}(F_{0m}(X)) = h^{1,0}(F_0(X')) = 0$$

we have that their Hodge numbers (as well as their decomposition on symmetric and antisymmetric part with respect to  $i_{F_m(X)}, i_{F(X')}$ ) also coincide. Moreover, we have the following properties of inclusions  $F_m(X) \hookrightarrow P^5$ ,  $F(X') \hookrightarrow P^5$ :  $\deg F_m(X) = \deg F(X') = 40$ ,  $\omega_{F_{0m}(X)} = \omega_{F_0(X')} = \mathcal{O}_{P^5}(3) \otimes \sigma$ . Nevertheless, some properties of  $F_m(X)$  and  $F(X')$  are different. For example, we have for  $X'$ :

- (a) There 12 conics passing through a generic point  $x \in X'$ ;

- (b)  $\xi|_{\bar{W}} : \bar{W} \rightarrow W$  is a six-sheeted covering;
- (c) image of  $F(X')$  in  $P^5$  is contained in the Plücker quadric (apparently this is not true for the image of  $F_m(X)$  in  $P^5$ );
- (d) There are no exceptional points and straight lines on  $F(X')$ .

[16] contains a conjecture that  $X$  and  $X'$  can be birationally isomorphic. Differences of properties of their Fano surfaces given above show that this is few likely. Nevertheless, there exists a Fano threefold  $X''$  ([1]) whose properties are in some sense "intermediate" between properties of  $X$  and  $X'$ .

Namely, let  $G_3$  be a  $G(2, 5) \cap P^3$ .  $X''$  is a two-sheeted covering of  $G_3$  with ramification in  $W = G_3 \cap \Omega$ , where  $\Omega$  is a quartic surface in  $P(\lambda^2(V))$ . There exists a variation in a smooth family of  $X$  in  $X''$ . Let  $K$  be a cone over  $G_4$  with vertex  $t$  and  $X_4 = K \cap \Omega$ . Then a section of  $X_4$  by a hyperplane which does not contain  $t$  is isomorphic to a  $X$  and a section of  $X_4$  by a hyperplane which contains  $t$  is isomorphic to a  $X''$ . From another side, some properties of  $X''$  are analogous to the ones of  $X'$ . For example,  $F_0(X'')$  (an analog of  $F_0(X')$ ) is isomorphic to the set of conics on  $G_3$  which are bitangent to  $W$ .

#### Section 4. Tangent bundle theorem.

Firstly we shall prove this theorem "locally". Recall that there exist maps  $F \rightarrow P_0 \rightarrow G_Z(2, M^*)$  and  $F \rightarrow P_Z(M^*)$ , so the corresponding tautological sheaves  $\tau_{2,m}$  and  $\tau_{3,m}$  are defined on  $F$ . We denote  $l = l_1 \cup l_2 \hookrightarrow F$ .

**Proposition 4.1.** Let  $c \in F - l$ . Then there exists a natural isomorphism  $\Omega_F(c) \rightarrow \tau_{2,m}^*(c)$ .

Proof of this proposition is a logically necessary step of the proof of the Tangent Bundle Theorem for  $F$ , because it implies the following

**Corollary 4.2.**  $F$  is smooth,  $\dim F = 2$ ,  $T$  is a regular bundle.  $\square$

Steps of the proof of Proposition 4.1 correspond to ones of the proof of Theorem 4.14 below. The proof of (4.14) is more complicated only because of necessity to consider exceptional straight lines  $l_1, l_2$ .

**Proof of 4.1.** We introduce the following notation. Let  $\alpha : P_{P(V^*)}(M) \rightarrow P(V^*)$  is the projectivization of  $M$  and  $O_\alpha(1)$  is the tautological sheaf on  $P_{P(V^*)}(M)$ . There exists an inclusion of sheaves  $M \rightarrow V_8 \otimes O$  on  $P(V^*)$  and hence maps

$$\begin{array}{ccccccc} P(M(V_4)) & & X & \hookrightarrow & G_4 & \hookrightarrow & G \\ \downarrow & & & & \downarrow & & \downarrow \\ P_{P(V^*)}(M) & \hookrightarrow & P_{P(V^*)}(V_8 \otimes O) & \xrightarrow{\text{pr}} & P(V_8) & \hookrightarrow & P(\lambda^2(V)) \end{array}$$

where vertical maps are inclusions and  $P_{P(V^*)}(V_8 \otimes O) = P(V^*) \times P(V_8)$ .

We denote by  $O_\alpha(1)$  the sheaf  $O_{P(\lambda^2(V))}(1)$  and its inverse images in all objects of this diagram, as well as in subvarieties of  $P(M(V_4))$  (for a generic  $V_4$ ).

Let  $c = (c, V_4) \in F - l$ . We consider diagrams

$$\begin{array}{ccc} c & \rightarrow & \pi(c) \\ \downarrow & & \downarrow \\ Q_G(V_4) & \rightarrow & P(M(V_4)) \end{array} \tag{4.3}$$

$$\begin{array}{ccc}
c & \rightarrow & X \\
\downarrow & & \downarrow \\
Q_G(V_4) & \rightarrow & G_4
\end{array} \tag{4.4}$$

According deformation theory,  $T_F(c) = H^0(c, N(c, X))$ , if

$$H^1(c, N(c, X)) = 0 \tag{4.5}$$

Later it will be shown that condition (4.5) is satisfied, hence we denote  $H^0(c, N(c, X))$  by  $T_F(c)$ . Analogous notations will be used in other cases where we shall use deformation theory. According Serre duality for  $c$  we have:

$$\begin{aligned}
H^0(c, N(c, X)) &= H^1(c, N(c, X)^* \otimes \omega_c)^* = H^1(c, N(c, X)^* \otimes \det N(c, X)^* \otimes \omega_c)^* \\
&= H^1(c, N(c, X) \otimes \omega_X)^*
\end{aligned}$$

and analogously  $H^1(c, N(c, X)) = H^0(c, N(c, X) \otimes \omega_X)^*$ , i.e.

$$\Omega_F(c) = H^1(c, N(c, X) \otimes \omega_X)^* \tag{4.6}$$

if  $H^0(c, N(c, X) \otimes \omega_X) = 0$ .

Since  $c \notin l$  we have that (4.3) is a fibred product, and hence

$$N(c, Q_G(V_4)) = N(\pi(c), P(M(V_4))) = O_\alpha(1)$$

Exact sequences of normal sheaves for both pairs of inclusions in (4.4) are the following:

$$0 \rightarrow N(c, X) \rightarrow N(c, G_4) \rightarrow N(X, G_4)|_c \rightarrow 0 \tag{4.7}$$

$$0 \rightarrow N(c, Q_G(V_4)) \rightarrow N(c, G_4) \rightarrow N(Q_G(V_4), G_4)|_c \rightarrow 0 \tag{4.8}$$

Let us consider the composite map

$$O_\alpha(1) = N(c, Q_G(V_4)) \rightarrow N(c, G_4) \rightarrow N(X, G_4)|_c = O_\alpha(2) \tag{4.9}$$

Further, let us consider a diagram

$$\begin{array}{ccccc}
c & \hookrightarrow & c \cup c' & \rightarrow & X \\
& & \downarrow & & \downarrow \\
& & Q_G(V_4) & \rightarrow & G_4
\end{array}$$

where  $c' = i_f(c)$ . We have  $c \cup c' = X \times_{G_4} Q_G(V_4)$ , so  $\tilde{N}(c \cup c', Q_G(V_4))|_c \rightarrow N(X, G_4)|_c$  is an isomorphism (here and below  $\tilde{N}(Y_1, Y_2)$  means — for an inclusion of varieties  $Y_1 \subset Y_2$  — a locally free sheaf on  $Y_2$  such that  $Y_1$  is the set of zeros of a section of it).

**(4.10).** There is a map  $N(c, Q_\Omega(V_4)) \rightarrow \tilde{N}(c \cup c', Q_G(V_4))|_c$  which is a multiplication by a section of a sheaf  $O_c(c \cap c') = O_\alpha(1)$  which corresponds to the divisor  $c \cap c'$  on  $c$ .

It is clear that  $N(Q_G(V_4), G_4) = \tau_{2,V}^*|_{Q_G(V_4)}$ . Multiplying (4.7) and (4.8) by  $\omega_X = O_\alpha(-1)$  we get

$$0 \rightarrow N(c, X) \otimes \omega_X \rightarrow N(c, G_4) \otimes O_\alpha(-1) \rightarrow O_\alpha(1) \rightarrow 0 \tag{4.11}$$

$$0 \rightarrow O \rightarrow N(c, G_4) \otimes O_\alpha(-1) \rightarrow \tau_{2,V}|_c \rightarrow 0 \quad (4.12)$$

According (4.10), the composite map

$$\phi_c : O \rightarrow N(c, G_4) \otimes O_\alpha(-1) \rightarrow O_\alpha(1)$$

corresponds to the divisor  $c \cap c'$  on  $c$ . Let us consider the long exact cohomology sequences for (4.11) and (4.12). Since  $c \notin l$ , we have  $\tau_{2,V}|_c = 2O_c(-1)$  and  $H^i(c, \tau_{2,V}|_c) = 0$ . This means that the long exact cohomology sequence for (4.11) is the following:

$$0 \rightarrow H^0(N(c, X) \otimes \omega_X) \rightarrow H^0(O_c) \xrightarrow{H^0(\phi_c)} H^0(O_\alpha(1)) \rightarrow \Omega_F(c) \rightarrow 0 \quad (4.13)$$

We have  $H^0(c, O_\alpha(1)) = \tau_{3,M}^*(c)$  (see (4.43) below) and  $H^0(\phi_c)$  is dual to an epimorphism

$$\tau_{3,M}(c) \rightarrow \tau_{3,M}(c)/\tau_{2,M}(c)$$

This means that  $H^0(\phi_c)$  is an inclusion,  $H^0(N(c, X) \otimes \omega_X) = 0$  and  $\Omega_F(c) = \tau_{2,M}^*(c)$   
 $\square$

**Theorem 4.14.** (The Tangent Bundle Theorem). There exists an exact sequence of sheaves on  $F$ :

$$0 \rightarrow \tau_{2,M}^* \rightarrow r^*(\Omega_{F_m}) \rightarrow i_*(O_l \oplus O_l) \rightarrow 0$$

where  $i$  is an inclusion  $l \hookrightarrow F$ .

**Proof.** Let us consider a Fano family  $D_F(c) \hookrightarrow F \times X$  defined as follows:  $(c, x) \in D_F(c) \iff x \in c$ , and let

$$\begin{array}{ccc} D_F(c) & \hookrightarrow & f \times X \\ & \searrow \beta & \downarrow \\ & & F \end{array}$$

be a diagram of natural projections on  $F$ , the vertical map is  $\alpha$ . According the deformation theory,  $r_F^*(T_{F_c}) = \beta_*(N(D_F(c), F \times X))$ , if  $\beta_{*1}(N(D_F(c), F \times X)) = 0$ . Like in (4.6), the relative Serre duality for  $\beta$  implies that

$$r_F^*(\Omega_{F_c}) = \beta_{*1}(N(D_F(c), F \times X) \otimes O_\alpha(-1))$$

if  $\beta_*(N(D_F(c), F \times X) \otimes O_\alpha(-1)) = 0$ .

Let us define the following subvarieties of  $P_F(M)$  (here  $(f, x)$  is an element of  $P_F(M) \subset F \times P(\lambda^2(V))$  and  $f = (c, V_4)$ ):

$$D_F(G) = \{(f, x) | x \in Q_G(V_4)\}$$

$$D_F(P^2) = P_F(\tau_{3,M}) = \{(f, x) | x \in \pi(c)\}$$

$$D_l(P^2) = P_l(\tau_{3,M}) = \{(f, x) | f \in l \text{ and } x \in \pi(c)\}$$

Recall that for  $f = (c, V_4) \in F$  we have  $Q_G(V_4) \cap \pi(c) = c$  if  $f \notin l$  and  $Q_G(V_4) \cap \pi(c) = \pi(c)$  if  $f \in l$ . Hence there is a diagram which is a fibred product:

$$\begin{array}{ccc} D_l(P^2) \cup D_F(c) & \rightarrow & D_F(P^2) \\ \downarrow & & \downarrow \\ D_F(G) & \rightarrow & P_F(M) \end{array} \quad (4.15)$$

Normal sheaves of inclusions of this diagram are the following. Since  $N(l, F) = O_\eta(1)$ , we have

$$N(D_l(P^2), D_F(P^2)) = O_\eta(1) \quad (4.16)$$

and since  $M/\tau_{3,M} = O_\pi(1)$  (recall that the projectivization map  $P_Z(M^*) \rightarrow Z$  is denoted by  $\pi$ ) we have

$$N(D_F(P^2), P_F(M)) = O_{(\alpha 1, \pi 1)} \quad (4.17)$$

Let us consider a diagram

$$\begin{array}{ccc} D_F(c) & \rightarrow & F \times X \\ \downarrow & & \downarrow \\ D_F(G) & \rightarrow & F \times G_4 \end{array} \quad (4.18)$$

We have

$$N(F \times X, F \times G_4) = O_\alpha(2) \quad (4.19)$$

$$\tilde{N}(D_F(G), F \times G_4) = O(1) \otimes \tau_{2,V}^* \quad (4.20)$$

where (4.20) is obtained by restriction to  $F \times G_4$  of the corresponding equality for a general flag variety  $D_{P(V_n^*)}(G) \subset P(V_n^*) \times G(k, V_n)$  defined as follows:

$$D_{P(V_n^*)}(G) = \{(V_{n-1}, g) | V_{n-1} \in P(V_n^*), g \in G(k, V_n), gV_n \subset V_{n-1}\}$$

We have  $N(D_{P(V_n^*)}(G), P(V_n^*) \times G(k, V_n)) = O_{P(V_n^*)}(1) \otimes \tau_{k,V_n}^*$ .

**Lemma 4.21.** Sing  $D_F(G) = S$ , where  $S \subset F \times Q_G(V_4)$  is defined as follows:

$$S = \{(f, x) | f \in l, x \in \{\gamma_1(f), \gamma_2(f)\}\}$$

( $f \in F, x \in Q_G(V_4)$ ).  $S$  is a double curve on  $D_F(G)$ .

**Proof.** Explicit calculations in coordinates of the equation of the tangent space of points of  $D_F(G)$ , and of the equation of the tangent cone of points of  $S$ . We omit these calculations.  $\square$

Let us consider the blowing up of (4.15) along  $S$  (recall that the blowing up along  $S$  of a variety  $Y \supset S$  is denoted by  $\tilde{Y} \xrightarrow{r} Y$ ):

$$\begin{array}{ccc} \widetilde{D_l(P^2)} \cup \widetilde{D_F(c)} & \rightarrow & \widetilde{D_F(P^2)} \\ \downarrow & & \downarrow \\ \widetilde{D_F(G)} & \rightarrow & \widetilde{P_F(M)} \end{array} \quad (4.22)$$

(4.22) is also a fibred product. (4.16), (4.17) imply that normal sheaves for inclusions are the following:

$$N(\widetilde{D_l(P^2)}, \widetilde{D_F(P^2)}) = O_{(\eta 1, r 1)} \quad (4.23)$$

$$N(\widetilde{D_F(P^2)}, \widetilde{P_F(M)}) = O(\alpha 1, \pi 1, r 1) \quad (4.24)$$

where  $O_r(1)$  is the tautological sheaf of the map  $r$ . (4.23), (4.24) imply

$$N(\widetilde{D_F(c)}, \widetilde{D_F(G)}) = O(\alpha 1, \pi 1, \eta - 1) \quad (4.25)$$

**Lemma 4.26.** Let us denote  $N = N(\widetilde{D_F(G)}, \widetilde{F \times G_4})$ . Then  $N$  appears in the exact sequence (4.31) below.

**Proof.** Let us consider a diagram of restrictions of blowings up on  $S$

$$\begin{array}{ccccccc} \tilde{S}_1 & \hookrightarrow & \tilde{S}_2 & \hookrightarrow & \tilde{S}_3 & \hookrightarrow & \tilde{S}_4 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S & = & S & = & S & = & S \end{array}$$

where  $\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4$  are inverse images of  $S$  in  $\widetilde{D_F(c)}, \widetilde{D_F(G)}, \widetilde{P_F(M)}, \widetilde{F \times G_4}$  respectively; projections (vertical arrows) are denoted by  $r_1, r_2, r_3, r_4$  respectively. For  $s \in S$   $r_1^{-1}(s) = P^1$ ,  $r_2^{-1}(s)$  is a 2-dimensional quadric,  $r_3^{-1}(s) = P^3$  which is the linear envelope of  $r_2^{-1}(s)$ , and  $r_4^{-1}(s) = P^4$ .

(4.20) and properties of blowings up imply that

$$\tilde{N}(\tilde{S}_3 \cup \widetilde{D_F(G)}, \widetilde{F \times G_4}) = O(1, r 1) \otimes \tau_{2,V}^* \quad (4.27)$$

If we have a diagram of inclusions

$$\begin{array}{ccc} Y' = Y_1 \cap Y_2 & \xrightarrow{i} & Y_1 \\ \downarrow & & \downarrow \\ Y_2 & \rightarrow & Y \end{array} \quad (4.28)$$

where all objects are smooth varieties,  $\text{codim}_Y Y_i = k$ ,  $\text{codim}_{Y_i} Y' = 1$  ( $i = 1, 2$ ), then there exists an exact sequence of sheaves on  $Y_1$

$$0 \rightarrow N(Y_1, Y) \rightarrow \tilde{N}(Y_1 \cup Y_2, Y)|_{Y_1} \rightarrow i_*(N(Y', Y_1) \otimes N(Y', Y_2)) \rightarrow 0 \quad (4.29)$$

We let in (4.28):  $Y = \widetilde{F \times G_4}$ ,  $Y_1 = \widetilde{D_F(G)}$ ,  $Y_2 = \tilde{S}_3$ ,  $Y' = \tilde{S}_2$ . In this case (4.29) is the following:

$$0 \rightarrow N \rightarrow O(1, r 1) \otimes \tau_{2,V}^* \rightarrow i_*(N(\tilde{S}_2, \tilde{S}_3) \otimes N(\tilde{S}_2, \widetilde{D_F(G)})) \rightarrow 0 \quad (4.30)$$

We have  $N(\tilde{S}_2, \widetilde{D_F(G)}) = O_r(-1)$ , and since for  $s \in S$   $r_2^{-1}(s)$  is a quadric in  $r_3^{-1}(s) = P^3$ , we have  $N(\tilde{S}_2, \tilde{S}_3) = O_r(2) \otimes r_2^*(L)$  for some sheaf  $L$  on  $S$ . It is possible to show that  $L = \tilde{N}(D_F(G), P_F(M))|_S = O(-1, \alpha 2)$ , but we do not need it. So, (4.30) can be written as follows:

$$0 \rightarrow N \rightarrow O(1, r 1) \otimes \tau_{2,V}^* \rightarrow i_*(O_r(1) \otimes r_2^*(L)) \rightarrow 0 \quad (4.31)$$

This is an exact sequence of sheaves on  $\widetilde{D_F(G)}$  that we need.  $\square$

Now we need an exact sequence of normal sheaves for a pair of inclusions  $\widetilde{D_F(c)} \hookrightarrow \widetilde{D_F(G)} \hookrightarrow \widetilde{F \times G_4}$ :

$$0 \rightarrow O(\alpha 1, \pi 1, \eta - 1) \rightarrow r^* N(D_F(c), F \times G_4) \otimes O_r(1) \rightarrow N|_{\widetilde{D_F(c)}} \rightarrow 0 \quad (4.32)$$

Let us restrict (4.31) to  $\widetilde{D_F(c)}$ . Since  $\tilde{S}_2$  and  $\widetilde{D_F(c)}$  are divisors in  $\widetilde{D_F(G)}$ ,  $\tilde{S}_2 \cap \widetilde{D_F(c)} = \tilde{S}_1$  and  $\text{codim}_{\widetilde{D_F(G)}} \tilde{S}_1 = 2$ , we have:  $\text{Tor}_1^{\widetilde{D_F(G)}}(i_* O_{\tilde{S}_2}, i_* O_{\widetilde{D_F(c)}}) = 0$ , and the restriction of (4.31) is the following:

$$0 \rightarrow N|_{\widetilde{D_F(c)}} \rightarrow O(1, r 1) \otimes \tau_{2,V}^* \rightarrow i_* O_r(1) \otimes L \rightarrow 0 \quad (4.33)$$

where  $i$  is an inclusion  $\tilde{S}_1 \rightarrow \widetilde{D_F(c)}$ .

Multiplying (4.32) and (4.33) by  $O_r(-1)$  we get

$$0 \rightarrow O(\alpha 1, \pi 1, \eta - 1, r - 1) \rightarrow r^* N(D_F(c), F \times G_4) \rightarrow N \otimes O_r(-1)|_{\widetilde{D_F(c)}} \rightarrow 0 \quad (4.34)$$

$$0 \rightarrow N \otimes O_r(-1)|_{\widetilde{D_F(c)}} \rightarrow O(1) \otimes \tau_{2,V}^* \rightarrow i_* L \rightarrow 0 \quad (4.35)$$

Now we consider direct images of the above sequences to  $D_F(c)$ . To do it, we consider an exact sequence of sheaves on  $\widetilde{D_F(c)}$ :

$$0 \rightarrow O \rightarrow O_r(-1) \rightarrow i_* O_r(-1)|_{\tilde{S}_1} \rightarrow 0$$

We get that  $r_* O_r(-1) = O$ ,  $r_{*i} O_r(-1) = 0$  for  $i > 0$ . Long exact  $r_*$ -direct images sequences of (4.34), (4.35) are the following (here we denote  $N' = r_*(N \otimes O_r(-1)|_{\widetilde{D_F(c)}})$ ):

$$0 \rightarrow O(\alpha 1, \pi 1, \eta - 1) \rightarrow N(D_F(c), F \times G_4) \rightarrow N' \rightarrow 0 \quad (4.36)$$

We get:  $r_{*1}(N \otimes O_r(-1)|_{\widetilde{D_F(c)}}) = 0$ , and

$$0 \rightarrow N' \rightarrow O(1) \otimes \tau_{2,V}^* \rightarrow i_* L \rightarrow 0 \quad (4.37)$$

An exact sequence of normal sheaves for a pair of inclusions  $D_F(c) \hookrightarrow F \times X \hookrightarrow F \times G_4$  is the following:

$$0 \rightarrow N(D_F(c), F \times X) \rightarrow N(D_F(c), F \times G_4) \rightarrow O_\alpha(2) \rightarrow 0 \quad (4.38)$$

Multiplying (4.36) — (4.38) by  $O_\alpha(-1)$  and taking the long exact  $\beta_*$ -direct image sequence we get (here we denote  $B = N(D_F(c), F \times G_4) \otimes O_\alpha(-1)$ ):

$$0 \rightarrow O(\pi 1, \eta - 1) \rightarrow \beta_*(B) \rightarrow \beta_*(N' \otimes O_\alpha(-1)) \rightarrow 0 \quad (4.39)$$

$$0 \rightarrow \beta_{*1}(B) \rightarrow \beta_{*1}(N' \otimes O_\alpha(-1)) \rightarrow 0 \quad (4.40)$$

$$0 \rightarrow \beta_*(N' \otimes O_\alpha(-1)) \rightarrow O(1) \otimes \beta_* \tau_{2,V} \rightarrow \beta_* i_*(L \otimes O_\alpha(-1)) \rightarrow \beta_{*1}(N' \otimes O_\alpha(-1)) \rightarrow$$

$$O(1) \otimes \beta_{*1} \tau_{2,V} \rightarrow \beta_{*1} i_*(L \otimes O_\alpha(-1)) \rightarrow 0 \quad (4.41)$$

$$\begin{aligned}
0 \rightarrow \beta_*(N(D_F(c), F \times X) \otimes O_\alpha(-1)) \rightarrow \beta_*(B) \rightarrow \beta_*O_\alpha(1) \rightarrow r_F^*(\Omega_{F_c}) \rightarrow \\
\beta_{*1}(B) \rightarrow \beta_{*1}O_\alpha(1) \rightarrow 0
\end{aligned} \tag{4.42}$$

It is easy to see that the restriction of  $\tau_{2,V}$  on a conic line  $c \subset G$  is equal to  $2O_c(-1)$  (resp.  $O_c \oplus O_c(-2)$ ) if  $c$  is not contained in a  $\alpha$ -plane (resp. if  $c$  is contained in a  $\alpha$ -plane). There exists the following property of the map  $\beta : D_F(c) \rightarrow F$ :

For  $f \in F$  we have:  $\beta^{-1}(f)$  is a conic line on an  $\alpha$ -plane  $\iff f \in l_2$ .

Hence, only for  $f \in l_2$  cohomology of a sheaf  $\tau_{2,V}|_{\beta^{-1}(f)}$  are non-zero. This means that  $\beta_*\tau_{2,V} = 0$  and  $\beta_{*1}\tau_{2,V}$  is a sheaf whose support is  $l_2$ , and it is invertible on  $l_2$ . (4.41) implies that  $\beta_*(N' \otimes O_\alpha(-1)) = 0$ , and (4.39) implies that  $\beta_*(B)$  is isomorphic to  $O(\pi_1, \eta - 1)$ .

Since we have a commutative diagram

$$\begin{array}{ccc}
S & \xrightarrow{i} & D_F(c) \\
\downarrow & & \downarrow \\
l & \xrightarrow{i} & F
\end{array}$$

where vertical maps are  $\beta_S, \beta$  respectively, we see that  $\beta_{*k}i_*(L \otimes O_\alpha(-1)) = i_*\beta_{S*k}(L \otimes O_\alpha(-1))$ . Since  $\beta_S$  is a two-sheeted covering, we have  $\beta_{*1}i_*(L \otimes O_\alpha(-1)) = 0$  and  $\beta_*i_*(L \otimes O_\alpha(-1))$  is a locally free sheaf of rank 2 on  $l$ . Let us consider a diagram

$$\begin{array}{ccc}
D_F(c) & \xrightarrow{i_c} & D_F(P^2) = P_F(\tau_{3,M}) \\
\searrow \beta & & \downarrow \\
& & F
\end{array}$$

where the vertical map is  $\alpha$ . Since for any  $f \in F$  we have an equality of sheaves on  $\alpha^{-1}(f)$ :  $O_{\alpha^{-1}(f)}(\beta^{-1}(f)) = O_\alpha(2)$ , we have globally on  $D_F(P^2)$ :  $O_{D_F(P^2)}(D_F(c)) = \alpha^*L_F \otimes O_\alpha(2)$ , where  $L_F$  is some invertible sheaf on  $F$ . There is an exact sequence of sheaves on  $D_F(P^2)$ :

$$0 \rightarrow L_F^{-1} \otimes O_\alpha(-1) \rightarrow O_\alpha(1) \rightarrow i_{c*}(O_\alpha(1)|_{D_F(c)}) \rightarrow 0$$

Considering its long exact  $\alpha_*$ -direct image sequence we see that

$$\beta_*O_\alpha(1) = \tau_{3,M}^* \tag{4.43}$$

and that  $\beta_{*1}(O_\alpha(1)) = 0$ .

Now let us consider a map  $i_B : \beta_*(B) \rightarrow \beta_*O_\alpha(1)$  which is isomorphic to  $O(\pi_1, \eta - 1) \rightarrow \tau_{3,M}^*$ . Its fibre at a point  $c = (c, V_4) \in F - l$  is  $i_B(c) = H^0(\phi_c)$  because of (4.13), i.e.  $i_B(c)$  is an inclusion  $(\tau_{3,M}(c)/\tau_{2,M}(c))^* \rightarrow \tau_{3,M}^*(c)$ . Tautological exact sequences on  $P_Z(M^*)$  and on  $G_Z(2, M^*)$  and an equality of sheaves on  $F$ :  $O_\eta(1) \otimes \det \tau_{2,M}^* = O$  imply an equality  $(\tau_{3,M}/\tau_{2,M})^* = O(\pi_1, \eta - 1)$ . So, there are 2 maps of sheaves

$$(\tau_{3,M}/\tau_{2,M})^* = O(\pi_1, \eta - 1) \rightarrow \tau_{3,M}^*$$

namely  $i_B$  and the natural inclusion  $(\tau_{3,M}/\tau_{2,M})^* \hookrightarrow \tau_{3,M}^*$  whose cokernel is  $\tau_{2,M}^*$ . For all  $f \in F - l$  their fibres at  $f$  coincide, hence the maps coincide as well. So, the exact sequence (4.42) (resp. (4.41), taking into consideration (4.40) ) can be rewritten as follows:

$$0 \rightarrow \tau_{2,M}^* \rightarrow r^*(\Omega_{F_c}) \rightarrow \beta_{*1}(B) \rightarrow 0 \quad (4.44)$$

respectively

$$0 \rightarrow \beta_* i_*(L \otimes O_\alpha(-1)) \rightarrow \beta_{*1}(B) \rightarrow O(1) \otimes \beta_{*1}\tau_{2,V} \rightarrow 0 \quad (4.45)$$

where  $\beta_* i_*(L \otimes O_\alpha(-1))$  is a locally free sheaf of rank 2 on  $l$  and  $O(1) \otimes \beta_{*1}\tau_{2,V}$  is an invertible sheaf on  $l_2$ .

Let us prove that  $O(1) \otimes \beta_{*1}\tau_{2,V} = \Omega_{l_2}$  and the composition map

$$r^*(\Omega_{F_c}) \rightarrow \beta_{*1}(B) \rightarrow O(1) \otimes \beta_{*1}\tau_{2,V}$$

coincides with the natural epimorphism  $r^*(\Omega_{F_c}) \rightarrow i_{2*}(\Omega_{l_2})$  associated to an inclusion  $i_2 : l_2 \rightarrow F_c$ .

We define respectively  $D_{l_2}(c)$ ,  $D_{l_2}(P^2)$ ,  $D_{l_2}(G) = D_{l_2}(P^2) \cup l_2 \times U^*$  as restrictions on  $l_2$  of  $D_F(c)$ ,  $D_F(P^2)$ ,  $D_F(G)$  respectively. We find  $\Omega_{l_2}$  by using deformation theory and a fact that  $l_2$  is a variety of  $\alpha$ -planes on  $G_4$ . Let

$$\begin{array}{ccc} D_{l_2}(P^2) & \hookrightarrow & l_2 \times G_4 \\ & \searrow & \downarrow \\ & & l_2 \end{array}$$

be the corresponding family (both maps to  $l_2$  are  $\alpha$ ); then we have:

$$T_{l_2} = \alpha_*(N(D_{l_2}(P^2), l_2 \times G_4)), \text{ if}$$

$$\alpha_{*1}(N(D_{l_2}(P^2), l_2 \times G_4)) = 0 \quad (4.46)$$

Serre duality implies that  $\Omega_{l_2} = \alpha_{*2}(N(D_{l_2}(P^2), l_2 \times G_4) \otimes \omega_{G_4})$ . Now we apply (4.29) to the diagram

$$\begin{array}{ccc} P_{l_2}(\tau_{2,M}) & \xrightarrow{i} & D_{l_2}(P^2) \\ \downarrow & & \downarrow \\ l_2 \times U^* & \rightarrow & l_2 \times G_4 \end{array}$$

We get an exact sequence of sheaves on  $D_{l_2}(P^2)$ :

$$0 \rightarrow N(D_{l_2}(P^2), l_2 \times G_4) \rightarrow O(1) \otimes \tau_{2,V}^* \rightarrow i_*(N_1 \otimes N_2) \rightarrow 0$$

where  $N_1 = N(P_{l_2}(\tau_{2,M}), D_{l_2}(P^2))$ ,  $N_2 = N(P_{l_2}(\tau_{2,M}), l_2 \times U^*)$ . We multiply it by  $\omega_{G_4} = O_\alpha(-3)$  and consider the long exact  $\alpha_*$ -direct image sequence. Since  $\forall x \in l_2$  we have  $\tau_{2,V}^*|_{\alpha^{-1}(x)} = O \oplus O_\alpha(1)$ , then  $\alpha_{*i}(O(1, \alpha-3) \otimes \tau_{2,V}^*)$  is 0 for  $i = 0, 1$  and one-dimensional for  $i = 2$ . Restriction of both sheaves  $N_1$  and  $N_2$  to  $\tau_{2,M}(x)$  (i.e. the fibre of the projection  $P_{l_2}(\tau_{2,M}) \rightarrow l_2$  at  $x$ ) is  $O_\alpha(1)$ , so  $N_1 \otimes N_2 \otimes O_\alpha(-3)$  at this fibre is equal to  $O_\alpha(-1)$ , and its cohomology is 0. This implies that the condition (4.46) is satisfied and

$$\Omega_{l_2} \rightarrow \alpha_{*2}(O(1, \alpha-3) \otimes \tau_{2,V}^*) \quad (4.47)$$

is an isomorphism. It is clear that a divisor  $D_{l_2}(c) \xrightarrow{i} D_{l_2}(P^2)$  corresponds to a sheaf  $O_\alpha(2)$ , so we consider an exact sequence on  $D_{l_2}(P^2)$ :

$$0 \rightarrow O_\alpha(-2) \rightarrow O \rightarrow i_*(O_{D_{l_2}(c)}) \rightarrow 0$$

We multiply it by  $O(1) \otimes \tau_{2,V} = O(1, \alpha - 1) \otimes \tau_{2,V}^*$  and consider a long exact  $\alpha_*$ -direct image sequence:

$$0 \rightarrow \alpha_{*1}(O(1) \otimes \tau_{2,V}|_{D_{l_2}(c)}) \rightarrow \alpha_{*2}(O(1, \alpha - 3) \otimes \tau_{2,V}^*) \rightarrow 0 \quad (4.48)$$

(4.47) and (4.48) imply existence of an isomorphism that we need. It is clear that it is compatible with the map  $r_F^*(\Omega_{F_c}) \rightarrow i_{2*}(\Omega_{l_2})$ . The kernel of this map is  $r^*(\Omega_{F_m})$ , where  $r : F \rightarrow F_m$ , and the exact sequences (4.44), (4.45) can be rewritten as follows:

$$0 \rightarrow \tau_{2,M}^* \rightarrow r^*(\Omega_{F_m}) \rightarrow \beta_* i_*(L \otimes O_\alpha(-1)) \rightarrow 0 \quad (4.49)$$

It is clear that  $r^*(\Omega_{F_m})|_l = 2O_l$ , and hence the restriction of (4.49) on  $l$  gives us an exact sequence

$$2O_l \rightarrow \beta_* i_*(L \otimes O_\alpha(-1))|_l \rightarrow 0 \quad (4.50)$$

Since  $\beta_* i_*(L \otimes O_\alpha(-1))|_l$  is a locally free sheaf of rank 2, (4.50) imply that it is isomorphic to  $2O_l$  (equality  $2O_l = \beta_* i_*(L \otimes O_\alpha(-1))$  can be proved also by means of explicit calculations). So, (4.49) can be rewritten as follows:

$$0 \rightarrow \tau_{2,M}^* \rightarrow r^*(\Omega_{F_m}) \rightarrow 2O_l \rightarrow 0 \quad (4.51)$$

Q.E.D.  $\square$

It is clear that (4.51) is isomorphic the the exact sequence associated to the divisor  $l \subset F$

$$0 \rightarrow O_\eta(-1) \rightarrow O \rightarrow i_*(O_l) \rightarrow 0$$

multiplied by  $r^*(\Omega_{F_m})$ . So,  $\tau_{2,M}^* = r^*(\Omega_{F_m}) \otimes O_\eta(-1)$ .

**Remark.** (4.51) can be used for a calculation of Chern classes of  $\Omega_F$ , because calculation of the class of  $\tau_{2,M}^*$  in  $K_0(F)$  is much easier than the one of  $\Omega_F$ .

## Section 5. Special intersection of 3 quadrics in $P^6$ .

In this section  $X$  will mean a special intersection of 3 quadrics in  $P^6$  (see below for its definition).

Firstly we continue to describe some properties of  $G(2, 5)$ . For a variety  $Y \subset P^n = P(V_{n+1})$  we denote by  $TP_Y(t) \subset P^n$  (resp.  $TV_Y(t) \subset V_{n+1}$ ) the projective (resp. vector) tangent space to  $Y$  at a point  $t \in Y$ , i.e.  $TP_Y(t) = P(TV_Y(t))$ .

For any  $n$   $G(2, V_n)$  is an intersection of quadrics in  $P(\lambda^2 V_n)$ , and the set of these quadrics is isomorphic to  $G(n-4, V_n)$ . Namely, for  $V_{n-4} \subset V_n$  the corresponding quadric is  $\text{Pl}(V_n/V_{n-4}) \in P(S^2 \lambda^2 V_n/V_{n-4})^*$ . For  $n = 5$  we denote the corresponding inclusion by  $q : P(V) \rightarrow P(S^2 \lambda^2 V^*)$ ; it is linear. All quadrics  $q(P(V))$  are cones whose vertex has dimension 3.

Let  $t \in G$ ,  $v \in P(V)$ . Then  $t \in \text{Sing } q(v) \iff v \in t_V$ .

**5.1.** Let  $TV_G(t) = V_7 \subset \lambda^2(V)$ . Then  $V_7$  is a member of exact sequences

$$0 \rightarrow \lambda^2 t_V \rightarrow V_7 \rightarrow t_V \otimes V/t_V \rightarrow 0$$

$$0 \rightarrow V_7 \rightarrow \lambda^2 V \rightarrow \lambda^2(V/t_V) \rightarrow 0$$

$\forall v \in t_V$  we have  $\text{Sing } q(v) \subset TP_G(t) \subset q(v)$ . The same inclusions are valid for projections from  $t$ .

**5.2.** Let  $c \subset G$  be a conic line such that  $\pi(c) \not\subset G$ . Then there exists the only  $V_4 \in P(V^*)$  such that  $v \in P(V_4) \iff \pi(c) \subset q(v)$ . It is easy to see that  $c \subset G(2, V_4)$ .

Let  $t \in G$ ; for any variety  $Y$  we denote by  $Y_t$  the projection of  $Y$  from  $t$ . Since  $\deg G = 5$ , we have:  $\deg G_t = 4$ , for  $v \in t_V$   $(q(v))_t$  is a quadric containing  $G_t$ . This means that  $G_t$  is a complete intersection of 2 quadrics. They generate a straight line  $P^1$  which is isomorphic to  $t_V$ , all quadrics belonging to this line are cones whose vertex is 2-dimensional, and they contain  $(TP_G(t))_t$ .

Let  $t \in G_4$  and  $t_V \cap U = \emptyset$ ; then  $(G_4)_t$  is a complete intersection of two quadrics in  $P^6$ , they generate a straight line  $P^1$  which is isomorphic to  $t_V$ , all quadrics belonging to this line are cones whose vertex is a point, they contain  $P_T^3 = (TP_G(t) \cap H_1 \cap H_2)_t$ , and their vertices are contained in  $P_T^3$  and form a normcubic  $\bar{c}_3$  in it. The set of straight lines on  $G_4$  passing through  $t$  is canonically isomorphic to  $\bar{c}_3$ ; a projection from  $t$  sends any such a line to the corresponding point of  $\bar{c}_3$ .

Let us recall now properties of  $X_8^3$  — a generic intersection of 3 quadrics in  $P^6 = P(V_7)$  (see [21], [28], [47], [50]). Let  $P^2$  be the linear envelope in  $P(S^2 V_7^*)$  of these 3 quadrics, and for  $h \in P^2$  let  $Q(h)$  be the corresponding quadric. The Hesse curve  $H_7 = H_7(X_8^3)$  is defined as follows:  $h \in H_7 \iff Q(h)$  is a cone. The set of planes on a non-singular 4-dimensional quadric  $Q$  has 2 connected components, we denote this set of components by  $\Gamma(Q)$ . Planes belonging to one component either coincide or their intersection is one point; intersection of planes belonging to different components is either empty or a straight line, and analogously for  $P^3$  lying on a cone over such a quadric. We denote a two-sheeted covering  $p = p_{H_7} : \hat{H}_7 \rightarrow H_7$  as follows:  $p^{-1}(h) = \Gamma(Q(h))$ .

For any plane curve  $H$  of degree  $n$  and its two-sheeted covering  $p : \hat{H} \rightarrow H$  whose involution is denoted by  $i_H : \hat{H} \rightarrow \hat{H}$ , we define a set  $\bar{S} = \bar{S}(p)$  of effective divisors on  $\hat{H}$  as follows:

$d \in \bar{S} \iff O_H(p_*(d)) = O_{P^2}(1)|_H \iff p_*(d)$  is a sum of  $n$  points  $H \cap l$  where  $l$  is a straight line in  $P^2$ .

$\bar{S}$  is non-connected,  $\bar{S} = S \cup S'$ , and if  $d \in \bar{S}$ ,  $d = \sum_{i=1}^n \hat{h}_i$  for  $\hat{h}_i \in \hat{H}$ , then  $d$  and  $d - \hat{h}_1 + i_H(\hat{h}_1)$  belong to different components of  $\bar{S}$ . Particularly, if  $n$  is even then  $S$  has an involution  $i_S : i_S(\sum \hat{h}_i) = \sum i_H(\hat{h}_i)$ .  $S$  is irreducible ([50]).

There exists an isomorphism  $\phi_7 : F(X_8^3) \rightarrow S = S(p_{H_7})$ , where  $F(X_8^3)$  is a set of conics in  $X_8^3$ .  $\phi_7$  is defined as follows. For a conic  $c \subset X_8^3$  we define firstly a straight line  $l(c) \subset P^2$  as follows:  $h \in l(c) \iff Q(h) \supset \pi(c)$ . For  $h \in l(c) \cap H_7$   $\pi(c)$  defines an element  $\gamma(c, h) \in \Gamma(Q(h)) = p^{-1}(h) \subset \hat{H}_7$ . We let

$$\phi_7(c) = \sum_{h \in l(c) \cap H_7} \gamma(c, h)$$

A fact that  $\phi_7$  is an isomorphism follows easily from the following proposition ([47]):

**5.3.** Let  $l \subset P(S^2V_7^*)$  be a straight line in a space of quadrics in  $P^6$ ,  $h_i \in l$  ( $i = 1, \dots, 7$ ) are such that  $Q(h_i)$  are cones. Then  $\cap_{h \in l} Q(h)$  contains  $2^6$  planes. Let  $\pi$  be such a plane. It defines a component in  $\Gamma(Q(h_i))$  denoted by  $\gamma(\pi, h_i)$ . Let us consider  $\gamma(\pi, h_i)$  as a function in  $i$ , it maps a set  $\{1, \dots, 7\}$  to  $\cup_{i=1}^7 \Gamma(Q(h_i))$ . Then these functions are different for different  $\pi$ , and values of 2 such functions are different at even numbers of elements of  $\{1, \dots, 7\}$ .

The composition map  $S \rightarrow \text{Div } \hat{H}_7 \rightarrow \text{Pic } \hat{H}_7$  (defined up to a shift) factors through the map  $\text{Pr } (\hat{H}_7/H_7) \hookrightarrow \text{Pic } \hat{H}_7$ .

**Theorem 5.4.** ([50]) The corresponding map  $\text{Alb } (S) \rightarrow \text{Pr } (\hat{H}_7/H_7)$  is an isogeny.  $\square$

**Theorem 5.5.** ([21]) There exists an isomorphism  $\text{Pr } (\hat{H}_7/H_7) \rightarrow J^3(X_8^3)$ .  $\square$

We denote the Abel - Jacobi map for conics by  $\Phi$ . We get a diagram where the left vertical map is  $2\Phi$ :

$$\begin{array}{ccc} \text{Alb } F & \xrightarrow{\text{Alb } (\phi_7)} & \text{Alb } S \\ \downarrow & & \downarrow \\ J^3(X_8^3) & \leftarrow & \text{Pr } (\hat{H}_7/H_7) \end{array}$$

**Theorem 5.6.** ([28]) It is commutative.  $\square$

Let us consider now properties of a threefold  $G_4 \cap \Omega$  which is smooth everywhere except one point  $t$  which is an ordinary double point. Recall that in this section it will be denoted by  $X$ . The set of conics on  $X$  will be denoted by  $F = F(X)$ . It is well-known that the projection of  $X$  from  $t$  is an intersection of 3 quadrics in  $P^6$ ; this fact is used for a proof of non-rationality of a "generic"  $G(2, 5) \cap H_1 \cap H_2 \cap \Omega$  ([26]).

**Lemma 5.7.** There exists a cone  $\Omega' \in P(S^2V_8^*)$  such that  $t$  is its vertex and such that  $X = G_4 \cap \Omega'$ .

**Proof.** We deprojectivize the map  $q : P(V) \rightarrow P(S^2\lambda^2V^*)$  and we restrict it to  $V_8$ . We get a map  $q : V \rightarrow S^2V_8$ . Let  $T : V \rightarrow V_8^*$  be a map defined as follows:  $T(v)$  is an equation of the polar line of  $q(v)$  with respect to  $t$ . Then  $\text{im } T \subset (V_8/TV_{G_4}(t))^*$  and  $\text{Ker } T$  is a set of points  $v \in V$  such that  $t \in \text{Sing } q(v)$ , i.e.  $\text{Ker } T = t_V$ , hence  $T$  is an epimorphism on  $(V_8/TV_{G_4}(t))^*$ . Since  $t$  is a singular point in  $G_4 \cap \Omega$ , we have  $TV_\Omega(t) \in (V_8/TV_{G_4}(t))^*$  and  $\exists v \in V$  such that  $T(v) = TV_\Omega(t)$ . This means that the equation of the tangent hyperplane at  $t$  of the quadric  $\Omega - q(v)$  is 0, i.e.  $\Omega' = \Omega - q(v)$  is the desired cone.  $\square$

Further we shall suppose that  $\Omega$  is such a cone. We see that  $X_t$  is an intersection of 3 quadrics:  $\Omega_t$  and  $(q(t_V))_t$ . As earlier, let  $P^2$  be the plane that they generate, and we denote by  $t_P$  the straight line that corresponds to quadrics  $(q(t_V))_t$ , or, in formal notations,  $h \in t_P \iff Q(h) \in (q(t_V)|_{H_1 \cap H_2})_t$ .

$t_P$  is a component of  $H_7(X_t)$ , another component is a curve of degree 6 denoted by  $H_6 = H_6(X)$ . It is clear that for a generic  $X$  it is non-singular, later we shall consider only these  $X$ . Let  $t_P \cap H_6 = \{R_1, \dots, R_6\}$ .  $Q(R_i)$  are ordinary cones, like all other quadrics corresponding to points of  $t_P$  and  $H_6$ . Hence there are a

two-sheeted non-ramified covering  $p = p_{H_6} : \hat{H}_6 \rightarrow H_6$  and a corresponding surface  $S = S(p_{H_6})$  together with an involution  $i_S$ . Vertices of  $Q(R_i)$  are projections of 6 straight lines on  $X$  passing through  $t$ , and they are the only singular points on  $X_t$ . These vertices are intersection points of  $\bar{c}_3$  and  $\Omega_t$  as well.

**Proposition 5.8.** There exists a birational isomorphism  $\phi_6 : F(X) \rightarrow S(p_{H_6})$ .

**Proof.** Let  $c = (c, V_4) \in F$ . If  $t \notin \pi(c)$  then the projection  $c_t \subset X_t$  is a conic. We denote by  $\chi$  a rational map  $(c, V_4) \mapsto c_t$  from  $F(X)$  to  $F(X_t)$ .  $c_t$  defines a straight line  $l(c) = l(c_t) \subset P^2$ . We denote  $l(c) \cap t_P = h_0(c)$ ,  $l(c) \cap H_6 = \{h_1(c), \dots, h_6(c)\}$ . If  $c \notin l_1 \cup l_2$ , then  $l(c) \neq t_P$  and (5.2) implies that  $Q(h_0(c)) = q(t_V \cap V_4)$ .

**Lemma 5.9.**  $\text{Sing } Q(h_0(c)) = P(M(V_4))_t \cap P_T^3$ .

**Proof.** This statement is formulated on  $G(2, V)$  as follows. For some given  $t \in G$ ,  $V_4 \subset V$  we set  $s = t_V \cap V_4$ . Then the linear envelope of  $t$  and  $P(\lambda^2(V_4) \cap TP_G(t))$  coincide with  $\text{Sing } q(s)$ . Really, let  $s_0, t_1$  (resp.  $s_0, v_1, v_2, v_3$ ) be a basis of  $t_V$  (resp.  $V_4$ ). Then (considering vector spaces) we have

a basis of  $t \subset \lambda^2 V$  is  $s_0 \wedge t_1$

a basis of  $TV_G(t)$  is  $s_0 \wedge t_1, s_0 \wedge v_i, t_1 \wedge v_i$

a basis of  $\lambda^2 V_4$  is  $s_0 \wedge v_i, v_i \wedge v_j$

and a basis of  $\text{Sing } q(s)$  is  $s_0 \wedge t_1, s_0 \wedge v_i$  (see (5.1);  $i, j = 1, 2, 3$ ). This implies the proposition for  $G$ . Intersecting it with  $H_1, H_2$  and projecting from  $t$  we get the desired.  $\square$

$\forall h \in t_P$  the quadric  $Q(h)$  contains  $P_T^3$ , we denote the corresponding element of  $\Gamma(Q(h))$  by  $\beta(h)$ .

**Lemma 5.10.**  $\beta(h_0(c)) \neq \gamma(c_t, h_0(c))$ .

**Proof.** Since  $\pi(c) \subset P(M(V_4))$ , we have:  $\pi(c_t)$  and  $(P_T^3)_{\text{Sing } Q(h_0(c))}$  have the empty intersection after the projection from  $\text{Sing } Q(h_0(c))$ .  $\square$

We define  $\phi_6(c) = \sum_{i=1}^6 \gamma(c_t, h_i(c))$ . (5.10) and (5.3) imply that at a generic point  $\phi_6$  is an inclusion. Since  $S$  is irreducible and  $\dim F = \dim S$  we have that  $\phi_6$  is a birational isomorphism.  $\square$

**Lemma 5.11.**  $\phi_6$  commutes with involutions  $i_F, i_S$ .

**Proof.** Let  $(c, V_4)$  and  $(c', V_4) \in F$  be 2 involutory conics. Then  $\pi(c) \cap \pi(c')$  is a straight line, hence (if  $t \notin P(M(V_4))$ , i.e. if  $V_4 \not\supset t_V$ )  $\pi(c_t) \cap \pi(c'_t)$  is also a straight line. For  $h \in l(c)$   $Q(h)$  contains  $c'_t$  and  $\pi(c_t) \cap \pi(c'_t)$ , i.e. a curve of degree 3 on  $\pi(c'_t)$  and hence the whole  $\pi(c'_t)$ , i.e.  $l(c) = l(c')$ ,  $h_i(c) = h_i(c')$ . The intersection of projections of  $\pi(c_t)$  and  $\pi(c'_t)$  from  $\text{Sing } Q(h_i(c))$  ( $i = 1, \dots, 6$ ) is a straight line, and if

$$\text{Sing } Q(h_i(c)) \notin P(M(V_4))_t$$

then  $\gamma(c_t, h_i(c)) \neq \gamma(c'_t, h_i(c'))$ . But else

$$P(M(V_4))_t \subset Q(h_i(c)) \Rightarrow \text{Sing } Q(h_0(c)) \in Q(h_i(c)).$$

If  $h_i(c) \notin \{R_1, \dots, R_6\}$  then  $h_i(c) \notin t_P$  and  $\text{Sing } Q(h_0(c))$  belongs to 3 basis quadrics of  $P^2$  and hence to  $X_t$ , i.e.  $h_0(c) \in \{R_1, \dots, R_6\}$ . This is not true at an open set, hence at this set  $c$  and  $c'$  are mapped to involutory elements of  $S$ , hence the last is true at all  $F$ .  $\square$

The above description of  $\phi_6$  does not indicate images of points  $(c, V_4) \in F$  for which  $V_4 \supset t_V$ , as well as images of straight lines  $l_1, l_2 \subset F$ . The set of spaces  $V_4$  which contain  $t_V$  is isomorphic to  $P_T^2 = P((V/t_V)^*) \hookrightarrow P(V^*)$ . For this  $V_4$   $Q_\Omega(V_4)$  is a cone in  $P(M(V_4))$  whose vertex is  $t$ , and  $Q_G(V_4)$  is a generic quadric containing  $t$ . Let  $(c, V_4) \in F$ ,  $V_4 \in P_T^2$  and

$$Q = T(\bar{\phi}(c)) = \pi(c) \cup \pi(c') \in \langle Q_\Omega(V_4), Q_G(V_4) \rangle$$

is a pair of planes. If  $Q \notin Q_\Omega(V_4)$ , then  $Q_\Omega(V_4) \cap Q_G(V_4) = c \cup c' = Q \cap Q_\Omega(V_4)$ . Let  $t \in \pi(c)$ , then  $Q_\Omega(V_4) \cap \pi(c)$  is a pair of straight lines passing through  $t$ , and  $Q_\Omega(V_4) \cap \pi(c')$  is a generic conic. There are 6 straight lines on  $X$  passing through  $t$ , i.e. there are only  $\binom{6}{2} = 15$  such spaces  $V_4$  while there are 30 conics. It is possible to prove (although we do not need it) that  $\phi_6$  maps these conics (as well as lines  $l_1, l_2 \subset F$ ) to 32 points  $S \cap ((p_{H_6})^{-1}(\sum_i R_i))$ .

Let now  $Q = Q_\Omega(V_4)$ . We denote the set of these  $V_4 \in P_T^2$  by  $K_0$ , i.e.  $K_0 = l_\Omega \cap P_T^2$ . We multiply an inclusion of vector spaces  $(t) \rightarrow V_8$  by  $O_{P_T^2}$ ; it is clear that the obtained map  $t \otimes O_{P_T^2} \rightarrow V_8 \otimes O_{P_T^2}$  factors through the inclusion  $M|_{P_T^2} \rightarrow V_8 \otimes O_{P_T^2}$ , and we denote  $\text{Coker}(t \otimes O_{P_T^2} \rightarrow M|_{P_T^2})$  by  $M_t$ . Since  $\Omega$  is a cone whose vertex is  $t$ , we have: a bundle of quadrics  $Q_\Omega$  restricted to  $P_T^2 \subset P(V^*)$  induces a bundle of quadrics  $Q_{\Omega_t}$  on  $(P_T^2, M_t)$  whose fibre at  $V_4 \in P_T^2$  is a projection  $Q_\Omega(V_4)$  from its vertex. It is clear that  $K_0 = D_2(Q_{\Omega_t})$ , hence  $K_0$  is a curve on  $P_T^2$ , and according to the Porteus formula ([7]) we have  $\deg K_0 = 2c_1(M_t^*) = 6$ .

Since  $K_0 \subset F_0(X)$ , there exists a two-sheeted non-ramified covering  $p_K : K \rightarrow K_0$  whose involution is denoted by  $i_K : K \rightarrow K$ , and  $p_K = p_F|_K$ ,  $i_K = i_F|_K$ . Involutory conics corresponding to points of  $K$  have 2 points of intersection, one of which is  $t$ . For a generic cone  $\Omega$  whose vertex is  $t$ , we have:  $K_0$  is non-singular and  $K$  is connected.

Let  $d : \tilde{F} \rightarrow F$  be the desingularization map and  $\tilde{K} = d^{-1}(K)$ . There exists an involution  $i_{\tilde{F}}$  on  $\tilde{F}$  such that  $i_F \circ d = d \circ i_{\tilde{F}}$ . There exist rational maps  $\tilde{\phi}_6 : \tilde{F} \rightarrow S$  and  $\tilde{\chi} : \tilde{F} \rightarrow F(X_t)$ .

**Proposition 5.12.**  $\tilde{K}$  is non-connected,  $\tilde{K} = K_1 \cup K_2$  and  $d_i = d|_{K_i} : K_i \rightarrow K$  are isomorphisms ( $i = 1, 2$ ). For  $f \in \tilde{K}$  we have:  $d(f)$  is a conic on  $X$  passing through  $t$ ,  $\tilde{\chi}(f)$  is a pair of straight lines on  $X_t$ , one of them is contained in a quadric  $Q_2 = P_T^3 \cap \Omega_t$ .

**Proof.** Since  $S$  is smooth and  $\tilde{K}$  is not a union of straight lines, we have that  $\tilde{\phi}_6(\tilde{K})$  is a curve on  $S$ . Let us find which points  $s \in S$  do not belong to  $\phi_6(F - K)$ . Let  $s = \sum_{i=1}^6 \hat{h}_i$ ,  $\hat{h}_i \in \hat{H}_6$ ,  $p_{h_6}(\hat{h}_i) = h_i \in H_6$ , and  $l$  a straight line in  $P^2$  such that  $\{h_i\} = l \cap H_6$ . We can suppose that  $l \neq t_P$  (this is true for almost all points  $s \in S$ ). We let  $h_0 = l \cap t_P$  and  $\hat{h}_0 \in \Gamma(Q(h_0))$ ,  $\hat{h}_0 \notin \beta(h_0)$ . According (5.3), there exists a plane  $\pi = \pi(s)$  on  $\cap_{h \in l} Q(h)$  that corresponds to the set  $\{\hat{h}_i\}$  ( $i = 0, \dots, 6$ ). Really, their images under the projection from  $\text{Sing } Q(h_0)$  must intersect namely by this way. Let  $\pi \cap P_T^3 = \emptyset$ ; then  $c = \pi \cap X_t$  is a conic. Let us consider the 2-dimensional cone containing  $c$  and having a vertex  $t \in P(V_8) \supset X$ . Since  $X$  is an intersection of quadrics, any straight line either intersects  $X$  at most at 2 points, or is contained in  $X$ . Since  $c \subset X_t$ , any ruling line of the above cone intersects  $X$  at one point (the other intersection point is  $t$ ). This point does not coincide with  $t$  because  $c \cap P_T^3 = \emptyset$ . So, the set of these intersection points is a conic, and  $s \in S$  is

its image under the map  $\phi_6$ . (If we chose a plane  $\pi$  on  $\cap_{h \in l} Q(h)$  that corresponds to a set  $\beta(h_0), \hat{h}_1, \dots, \hat{h}_6$ , then  $\pi \cap P_T^3$  is one point, and instead of a conic we should get a normcubic on  $X$  passing through  $t$ ).

So, if  $s = \tilde{\phi}_6(c)$  for  $c \in \tilde{K}$ , then  $\pi(s) \cap P_T^3 = P^1$ . It is clear that  $\pi(s) = \pi(\tilde{\chi}(c))$  and  $\forall h \in P^2 - l(c)$  we have  $\tilde{\chi}(c) = \pi(s) \cap Q(h)$ . Choosing  $h \in t_P$  we get that  $\tilde{\chi}(c) \supset \pi(s) \cap P_T^3 = P^1$ . This means that  $\tilde{\chi}(c)$  is a pair of straight lines, one of which lies on a quadric  $Q_2 = \Omega_t \cap P_T^3$ , and other intersects it.

Inversely, let  $l_X$  be any straight line on  $X_t$  that intersects  $Q_2$ . There are 2 ruling lines  $u_1, u_2$  of  $Q_2$  that pass through a point  $l_X \cap Q_2$ , hence we get 2 conics  $c_i(l_X) = l_X \cup u_i$  ( $i = 1, 2$ ). We can associate them:

- (a) 2 different straight lines  $l(c_i(l_X)) \subset P^2$ ;
  - (b) points  $h_{ij} = h_j(c_i(l_X)) \in H_6$ ;
  - (c) elements  $\gamma(c_i(l_X), h_{ij}) \in \hat{H}_6$
- such that the element

$$s_i(l_X) = \sum_{j=1}^6 \gamma(c_i(l_X), h_{ij}) \in S$$

and  $\pi(s_i(l_X)) \cap P_T^3 = u_i$ . Let us find  $\tilde{\phi}_6^{-1}(s_i(l_X))$ . A cone over  $c_i(l_X)$  having a vertex  $t$  in  $P(V_8)$  is a plane  $\langle l_X, t \rangle$ , and each straight line in this plane passing through  $t$  meets  $X$  exactly at one point (except  $t$ ). Further, one straight line in this plane (it is the inverse image of  $l_X \cap Q_2$  under the projection from  $t$ ) is tangential to  $X$  at  $t$ . So,  $\langle l_X, t \rangle \cap X$  is a conic  $c_X(l_X) \in K$ , and  $\tilde{\phi}_6^{-1}(s_i(l_X)) \in d^{-1}(c_X(l_X))$ . This means that  $\forall c \in K \quad d^{-1}(c)$  is 2 points, i.e.  $\tilde{K} \rightarrow K$  is a non-ramified two-sheeted covering, and for  $c \in \tilde{K}$  we have  $\tilde{\chi}(c) = d(c)_t \cup u(c)$ , where  $u(c)$  is a ruling line of  $Q_2$ . Since there are 2 systems of rulings on  $Q_2$ , we have:  $\tilde{K} = K_1 \cup K_2$  is non-connected, namely  $c \in K_i$  iff  $u(c)$  is a ruling line of type  $i$ . It is clear that restrictions  $d_i = d|_{K_i} : K_i \rightarrow K$  are isomorphisms ( $i = 1, 2$ ).  $\square$

**Remark.** Let  $\tilde{r} : \tilde{F} \rightarrow \tilde{F}_m$  be a map of blowing down of  $l_1 \cup l_2 \subset \tilde{F}$  to points. There exists an isomorphism  $i : \tilde{F}_m \rightarrow S$  such that  $\tilde{\phi}_6 = i \circ \tilde{r}$ .

Let us denote the isomorphism  $d_2^{-1} \circ d_1 : K_1 \rightarrow K_2$  by  $\alpha$ , and let  $i_{K_1} = d_1^{-1} \circ i_K \circ d_1$  be an involution of  $K_1$ .

**Lemma 5.13.** For  $c \in K_1$  we have  $i_{\tilde{F}}(c) = \alpha(i_{K_1}(c))$ .

**Proof.**  $i_{\tilde{F}}(c)$  is equal to  $i_{K_1}(c)$  or  $\alpha(i_{K_1}(c))$ . We have: the intersection of  $\tilde{\chi}(c)$  and  $\tilde{\chi}(i_{K_1}(c))$  is 1 point, while  $\forall f \in \tilde{F}$  the intersection of  $\tilde{\chi}(f)$  and  $\tilde{\chi}(i_{\tilde{F}}(f))$  is 2 points.  $\square$

We denote by  $\tilde{X}$  the desingularization of  $X$ . Any point of  $\tilde{F}$  defines a conic on  $\tilde{X}$  by an obvious way (for  $c \in \tilde{F}$  the corresponding conic is the inverse image of  $\tilde{\chi}(c)$  under the desingularization map  $\tilde{X} \rightarrow X$ ). So, there exists the Abel - Jacobi map  $\Phi : \tilde{F} \rightarrow J^3(\tilde{X})$ .

The next 3 propositions are analogs for  $X_t$  of the theorems (5.4) - (5.6) for a generic  $X_8^3$ , and their proofs follow the proofs of these theorems.

**Proposition 5.14.** The map  $\text{Alb } S(p_{H_6}) \rightarrow \text{Pr } \hat{H}_6/H_6$  is an isogeny.

**Proof.** [50], Proposition 1.3.3 implies that this map is an epimorphism, hence it is sufficient to prove that  $h^{1,0}(S) \leq 9$ . The proof of ([50], Proposition 3.4.4) implies that:

(1) There exist plane curves  $C_i$  of degree 6 ( $i = 0, \dots, 6$ ), their non-ramified two-sheeted coverings  $p_i : \hat{C}_i \rightarrow C_i$  and points  $P_1, \dots, P_6 \in P^2$  in a general position such that:

- (a)  $p_0 = p_{H_6}$ ;
- (b)  $\text{Sing } C_i = \{P_1, \dots, P_i\}$  and these points are ordinary double points on  $C_i$ ;
- (2)  $\forall i = 0, \dots, 5$  there exists a commutative diagram

$$\begin{array}{ccc} \hat{D}_i & \rightarrow & D_i \\ & \searrow & \downarrow \\ & & T_i \end{array}$$

where  $T_i$  is an open subset in  $P^1$ ,  $D_i$  is a smooth surface, the fibre of the covering  $\hat{D}_i \rightarrow D_i$  at a point  $t \in T_i$  is a non-ramified two-sheeted covering of smooth curves  $(\hat{D}_i)_t \rightarrow (D_i)_t$  for all  $t$  except  $t = 0$ , where  $(D_i)_t$  has one double point and  $(\hat{D}_i)_t$  has 2 double points which are its inverse images. Further, for  $t = t_0$  the above fibre is isomorphic to the normalization of the covering  $p_i$  denoted by  $\tilde{p}_i : \tilde{\hat{C}}_i \rightarrow \tilde{C}_i$ , and for  $t = 0$  it is isomorphic to the blowing up of the covering  $p_{i+1}$  at  $P_1 \cup \dots \cup P_i$  for  $C_{i+1}$  and their  $p_{i+1}$ -inverse images for  $\hat{C}_{i+1}$ .

(3) There exist maps  $D_i \rightarrow T_i \times P^2$  over  $T_i$  whose fibres at points  $t_0$  and  $0 \in T_i$  are ordinary maps  $\tilde{C}_i \rightarrow C_i \hookrightarrow P^2$  and  $(\tilde{C}_{i+1})_{P_1 \cup \dots \cup P_i} \rightarrow C_{i+1} \hookrightarrow P^2$ .

Some properties of  $C_i$  and coverings  $D_i \rightarrow T_i$  of purely technical nature are omitted, their complete list is given in [50], 2.2.1.

These conditions and ([50], 2.2.7) imply that

$$\forall i \quad h^{1,0}(S(\tilde{p}_i)) \leq h^{1,0}(S(\tilde{p}_{i+1})) + 1$$

If points  $P_1, \dots, P_6$  do not belong to a conic then the linear system of the sheaf  $\mathcal{O}_{P^2}(1)|_{\tilde{C}_6}$  is non-special. ([50], 1.3.8) implies that  $h^{1,0}(S(\tilde{p}_6)) = g(\tilde{C}_6) - 1 = 3$  and hence  $h^{1,0}(S(p_{H_6})) \leq 9$ .  $\square$

**Proposition 5.15.** There exists an isomorphism  $I : \text{Pr } \hat{H}_6/H_6 \rightarrow J^3(\tilde{X})$ .

**Proof.** Let  $\hat{h} \in \hat{H}_6$  and  $p(\hat{h}) = h$ . Let us consider the cone  $Q(h)$ . The set of spaces  $P^3$  of type  $\hat{h}$  on  $Q(h)$  is isomorphic to  $P^2$ , so for all these  $P^3$  curves  $P^3 \cap X$  are birationally equivalent. We denote such a curve by  $C_4(\hat{h})$ , they can be lifted on  $\tilde{X}$  and these lifts define a cylinder map  $\Phi : H_1(\hat{H}_6, \mathbb{Z}) \rightarrow H_3(\tilde{X}, \mathbb{Z})$  and an Abel - Jacobi map  $\hat{I} : J(\hat{H}_6) \rightarrow J^3(\tilde{X})$ .

Let us choose a generic straight line  $l \subset X$ ;  $\forall \hat{h} \in \hat{H}_6$  there exists the only  $P^3$  of type  $\hat{h}$  such that  $P^3 \supset l$ . We have

$$P^3 \cap X = C_4(\hat{h}) = l \cup C_3(\hat{h}, l)$$

where  $C_3(\hat{h}, l)$  is a normcubic in  $P^3$  meeting  $l$  at 2 points. A system of curves  $C_3$  defines the same cylinder map  $\Phi : H_1(\hat{H}_6, \mathbb{Z}) \rightarrow H_3(\tilde{X}, \mathbb{Z})$ . Let us apply a formula ([20], p. 44) to  $\Phi$ :

$$\forall \gamma_1, \gamma_2 \in H_1(\hat{H}_6, \mathbb{Z}) \quad \langle \Phi(\gamma_1), \Phi(\gamma_2) \rangle = k \langle \gamma_1, \gamma_2 \rangle + \langle i(\gamma_1), \gamma_2 \rangle$$

where  $k \in \mathbb{Z}$  and  $i \in \text{End } H_1(\hat{H}_6, \mathbb{Z})$  corresponds to a correspondence  $C$  on  $\hat{H}_6 \times \hat{H}_6$  defined as follows:  $(\hat{h}_1, \hat{h}_2) \in C \iff C_3(\hat{h}_1, l) \cap C_3(\hat{h}_2, l) \neq \emptyset$ . Further, [20], p. 96 implies that  $i = H_1(i_H)$ .

From another side, we have  $g(H_6) = 10$ ,  $h^{2,1}(\tilde{X}) = 9$  (because  $h^{2,1}(X) = 10$  and because  $\tilde{X}$  is a blowing up of a degenerated fibre of a Lefschetz pencil). This implies that  $\dim \text{Ker } \Phi \geq h_1(\hat{H}_6) - h_3(\tilde{X}) = 20$ , i.e. for a 20-dimensional space of cycles  $\gamma$  we have  $i(\gamma) + k\gamma = 0$ . This can be only for  $k = -1$ , hence  $\text{Ker } \Phi = J(H_6)$  and  $\text{Pr } \hat{H}_6/H_6 \rightarrow J^3(\tilde{X})$  is an isomorphism.  $\square$

**Proposition 5.16.** The diagram

$$\begin{array}{ccc} \text{Alb } (\tilde{F}) & \xrightarrow{\text{Alb } (\tilde{\phi}_6)} & \text{Alb } (S) \\ \downarrow & & \downarrow \\ J^3(\tilde{X}) & \xleftarrow{I} & \text{Pr } (\hat{H}_6/H_6) \end{array}$$

where the left vertical map is  $2\Phi$ , is commutative.

**Proof.** Let  $c \in \tilde{F}$ ,  $l(c) \cap t_P = h_0$ ,  $l(c) \cap H_6 = \{h_1, \dots, h_6\}$  and  $\gamma(c_t, h_i) = \hat{h}_i$ . Then  $\phi_6(c) = \sum_{i=1}^6 \hat{h}_i$  and it is sufficient to prove that codimension 2 cycles  $2c - \sum_{i=1}^6 C_4(\hat{h}_i)$  on  $\tilde{X}$  are linearly equivalent for all  $c \in \tilde{F}$ . Let  $X_4 = \cap_{h \in l(c)} Q(h)$  be an intersection of 2 quadrics and  $P_i^3$  the only  $P^3$  of type  $\hat{h}_i$  containing  $\pi(c_t)$ . Then  $X_4 \cap P_i^3$  is a quadric containing  $\pi(c_t)$ , i.e.  $X_4 \cap P_i^3$  is a pair of planes  $\pi(c_t) \cup L_i$ . According ([47], Section 3, Lemma 6) there is a linear equivalence ( $\equiv$ ) on  $X_4$  of 2 cycles:

$$\sum_{i=0}^6 L_i + 5\pi(c_t) \equiv 3(P^4 \cap X_4)$$

Intersecting this equivalence with  $X$  we get:

$$\sum_{i=0}^6 L_i \cap X + 6c \equiv 2c - (c + L_0 \cap X) + \text{const}$$

But  $c + L_0 \cap X = \text{const}$ , because  $c \cup (L_0 \cap X) = P_0^3 \cap X$  and  $P_0^3$  varies along  $t_P = P^1$ . Equality  $C_4(\hat{h}_i) = (L_i \cap X) \cup c$  implies that  $\sum_{i=1}^6 C_4(\hat{h}_i) - 2c \equiv \text{const}$ .  $\square$

## Section 6. Abel - Jacobi map of $F(X)$ is an isogeny.

Here  $X$  will mean again a Fano threefold of genus 6. Let us prove that  $h^{1,0}(F(X)) = \dim J^3(X) = 10$ . We consider  $X'_4 = G \cap H_1 \cap \Omega$  and a Lefschetz pencil  $f' : X'_4 \rightarrow P^1$ , we restrict it to a neighbourhood of zero  $\Delta \subset P^1$ , so the only singular fibre  $f'^{-1}(t)$ ,  $t \in \Delta$ , is a threefold  $X_0 = f'^{-1}(0)$  having the only double point. We denote  $F'_3 = \cup_{t \in \Delta} F(f'^{-1}(t))$ , and  $f'_F : F'_3 \rightarrow \Delta$  is the projection.

**Lemma 6.1.** For a generic  $f' : X'_4 \rightarrow P^1$   $F'_3$  is non-singular.

**Proof.** Non-singularity of  $F'_3$  is not obvious only at points of  $K = F(X_0) \subset F'_3$ . For a given  $X_0$  it is possible to choose  $X'_4 \subset P^8 = P(H_1)$  such that  $X_0 = X'_4 \cap P^7$ ,

where  $P^7 \subset TP_{X'_4}(t)$  and such that all points of  $K$  are non-singular on  $F(X'_4)$  — the variety of conics on  $X'_4$ . A generic  $P^6 \subset P^7$  defines a Lefschetz pencil  $f' : X'_4 \rightarrow P^1$ ; we denote by  $U(P^6)$  a set of conics  $c \subset P^8$  such that  $\dim(\pi(c) \cap P^6) \geq 1$ . We have  $F'_3 = U(P^6) \cap F(X'_4)$ . Let  $c \in K$ . Let us look which conditions must be imposed on  $P^6 \subset P^7$  in order to achieve non-singularity of  $F'_3 = F'_3(P^6)$  at points of  $c$ . For any variety  $Y \ni c$  we denote  $TV_Y(c)$  by  $\bar{Y}$ , and we denote by  $U_7$  (resp.  $U_8$ ) varieties of conics in  $P^7$  (resp.  $P^8$ ). We get a diagram of tangent spaces at  $c$ :

$$\begin{array}{ccccc} \overline{F(X_0)} & & \hookrightarrow & & \overline{F(X'_4)} \\ \downarrow & & & & \downarrow \\ \bar{U}_7 & \hookrightarrow & \overline{U(P^6)} & \hookrightarrow & \bar{U}_8 \end{array}$$

(vertical maps are inclusions). Here we have:  $\overline{F(X_0)} = \overline{F(X'_4)} \cap \bar{U}_7$ ,  $\text{codim}_{\bar{U}_8} \bar{U}_7 = 3$ ,  $\dim \overline{F(X'_4)} = 5$ ,  $\dim \overline{F(X_0)} = 3$ , because  $c$  is a smooth point on  $F(X_0)$ . Condition of smoothness of  $F'_3(P^6)$  at  $c$  is equivalent to a condition that  $\dim \overline{U(P^6)} \cap \overline{F(X'_4)} = 3$ , i.e. that  $\overline{U(P^6)} \not\subset \langle \overline{F(X'_4)}, \bar{U}_7 \rangle$ . But  $\overline{U(P^6)} \in P(\bar{U}_8/\bar{U}_7) = P^2$ , and a map  $P^6 \mapsto P(\overline{U(P^6)}/\bar{U}_7) \in P^2$  is a linear projection from  $P^{7*}$  to  $P^2$  whose center  $P^4 \subset P^{7*}$  is dual to  $\pi(c) \subset P^7$ . The inverse image of  $\langle \overline{F(X'_4)}, \bar{U}_7 \rangle$  under this projection is a hyperplane in  $P^{7*}$  containing  $P^4$ , i.e. the set of  $P^6 \subset P^7$  which contain a fixed point in  $\pi(c)$ . It is easy to see that this point is  $t$ . So, for a non-singularity of  $F'_3$  it is sufficient to choose a generic  $P^6$  which does not contain  $t$ .  $\square$

It is clear that for any  $X$  we can choose  $f' : X'_4 \rightarrow P^1$  such that:

1.  $X = f'^{-1}(t'_0)$  for  $t'_0 \in \Delta$ ;
2. The special fibre satisfies properties of general position of Section 5 (smoothness of  $H_6, K_0$ );
3.  $F'_3$  is non-singular.

**6.2.** To transform  $f'$ ,  $f'_F$  to a desired degeneration we use a standard method (see, for example, [30]). Let us consider blowings up of  $X'_4$  and of  $F'_3$  along singular points of the fibre over 0, we denote the corresponding maps by  $\tilde{f}' : \tilde{X}'_4 \rightarrow \Delta$ ,  $\tilde{f}'_F : \tilde{F}'_3 \rightarrow \Delta$ . Their fibres at 0 are  $\tilde{X}_0 \cup P^3$ ,  $\widetilde{F(X_0)} \cup R$  respectively, where  $P^3$ ,  $R$  are exceptional divisors of blowing up of  $X'_4$  and  $F'_3$ , and  $R$  is a fibration over  $K = K(F(X_0))$  whose fibres are isomorphic to  $P^1$ . It is clear that  $\tilde{X}_0 \cap P^3 = Q$  is a non-singular quadric in  $P^3$  and  $\widetilde{F(X_0)} \cap R = K_1 \cup K_2$ . Components of the fibre at 0 of  $P^3$  and  $R$  are double, that's why we consider a map  $\phi_2 : \Delta \rightarrow \Delta$ :  $t \mapsto t^2$  and a diagram which is a fibred product:

$$\begin{array}{ccc} X_4 & \rightarrow & \tilde{X}'_4 \\ \downarrow & & \downarrow \\ \Delta & \xrightarrow{\phi_2} & \Delta \end{array}$$

where the right vertical map is  $\tilde{f}'$ . We denote the left vertical map by  $f$ . The fibre of  $f$  at  $t_0 = (t'_0)^{\frac{1}{2}}$  (resp. at  $t = 0$ ) is  $X$  (resp.  $X(0) = \tilde{X}_0 \cup \bar{P}_3$ ), where  $\bar{P}_3$  is a two-sheeted covering of  $P^3$  ramified at  $Q$ , so  $\tilde{X}_0 \cap \bar{P}_3 = Q$ , and both components have multiplicity 1. Analogously we define a map  $f_F : F_3 \rightarrow \Delta$  having a singular fibre  $F(0) = F \cup \bar{R}$ , where  $F = \widetilde{F(X_0)}$ , and  $\bar{R} \rightarrow R$  is a two-sheeted covering ramified at

$K_1 \cup K_2$ , so  $F \cap \bar{R} = K_1 \cup K_2$ , and composition maps  $K_i \hookrightarrow \bar{R} \rightarrow R \rightarrow K$  coincide with  $d_i$  ( $i = 1, 2$ ).

Clemens - Schmidt exact sequence ([8], [40]) for  $f_F$  is the following:

$$0 \rightarrow H^1(F(0)) \xrightarrow{\nu} H^1(F(X)) \xrightarrow{N} H^1(F(X)) \quad (6.3)$$

There exists a mixed Hodge structure on  $H^1(F(0))$ ,  $H^1(F(X))$ , and  $\nu$  and  $N$  are strict morphisms with respect to these structures of degrees (0,0) and (-1, -1) respectively. We shall need only weight filtrations; their definition in our case is the following ([8]): for  $H^1(F(X))$  we have

$$0 \subset W_0 \subset W_1 \subset W_2 = H^1(F(X))$$

where  $W_0 = \text{im } N$ ,  $W_1 = \text{Ker } N$ . For  $H^0(F(0))$  we have

$$0 \subset W_0 \subset W_1 = H^1(F(0))$$

where  $W_0 = H^1(\Pi(F(0)))$  and  $\Pi(F(0))$  is a CW-complex whose vertices correspond to components of  $F(0)$  and edges correspond to irreducible components of intersections of these components. We have

$$W_1/W_0 = \text{Ker } H^1(F) \oplus H^1(\bar{R}) \rightarrow H^1(F \cap \bar{R})$$

for a natural restriction map. In our case  $\Pi(F(0)) = S^1$  (2 vertices, 2 edges), i.e.  $\dim W_0(H^1(F(0))) = 1$ . Exactness of (6.3) implies that  $\dim W_2/W_1 = \dim W_0(H^1(F(X))) = \dim W_0(H^1(F(0))) = 1$  and  $W_1/W_0$  for  $H^1(F(X))$  and  $H^1(F(0))$  coincide. So, the equality  $h^1(F(X)) = 20$  is a corollary of

$$\dim \text{Ker } H^1(F) \oplus H^1(\bar{R}) \rightarrow H^1(F \cap \bar{R}) = 18 \quad (6.4)$$

So, we have to prove (6.4).

**Lemma 6.5.** Let us identify  $K_1$  and  $K_2$  via  $\alpha$ . Then (6.4) follows from the following statement: the image of a restriction map

$$H^1(F) \rightarrow H^1(K_1) \oplus H^1(K_2) \quad (6.6)$$

is contained in the diagonal.

**Proof.** Since  $\bar{R} \rightarrow K$  is a fibration with a fibre  $P^1$ , we have:  $H^1(K) \rightarrow H^1(\bar{R})$  is an isomorphism. This means that the restriction map  $H^1(\bar{R}) \rightarrow H^1(K_1) \oplus H^1(K_2)$  is an inclusion, and its image is the diagonal. According (5.14),  $h^1(F) = 18$ , this implies the lemma.  $\square$

**Lemma 6.7.** (6.6) is equivalent to the following condition

$$\forall h \in H^1(F) \quad h|_K = i_K^*(i_F^*(h)|_K) \quad (6.8)$$

(here and below  $K = K_1$ ).

**Proof.** Follows from commutativity of the diagram

$$\begin{array}{ccccc} K_1 & \xrightarrow{i_K} & K_1 & \xrightarrow{\alpha} & K_2 \\ \downarrow & & & & \downarrow \\ F & \xrightarrow{i_F} & & & F \end{array}$$

(vertical maps are natural inclusions).  $\square$

There exists a composite map defined up to a shift

$$f : F \xrightarrow{\tilde{\phi}_6} S \rightarrow \text{Pr } \hat{H}^6/H^6$$

**Lemma 6.9.**  $f \circ i_F = -f$  (up to a shift).

**Proof.** For  $c \in F$  let  $f(c) = \sum_{i=1}^6 \hat{h}_i$  for  $\hat{h}^i \in \hat{H}_6$ ; then  $f(i_F(c)) = \sum_{i=1}^6 i_H(\hat{h}_i)$ . But  $O_{\hat{H}_6}(\sum_{i=1}^6 (\hat{h}_i + i_H(\hat{h}_i))) = O_{P^2}(1)|_{\hat{H}_6}$ , i.e.  $\sum \hat{h}_i \equiv -\sum i_H(\hat{h}_i) + \text{const}$ .  $\square$

This means that  $H^1(i_F)$  is a multiplication by  $-1$ . This is equivalent to the following:  $H^1(F_0) = 0$ , where  $F_0 = \widetilde{F_0(X_0)}$ .

Let  $E_i \hookrightarrow H^1(K)$  ( $i = \pm 1$ ) be eigenspaces of  $H^1(i_K)$  with eigenvalues  $i$ , then  $E_1 = H^1(K)$ ,  $E_{-1} = H^1(\text{Pr } K/K_0)$ , and (6.8) is equivalent to the following:

The image of the restriction map  $H^1(F) \rightarrow H^1(K)$  is contained in  $E_{-1}$ , or, which is the same, the composition map  $\text{Alb } K_0 \rightarrow \text{Alb } K \rightarrow \text{Alb } F$  is 0.

Taking into consideration (5.15) and (5.16), it is sufficient to prove that the map  $\text{Alb } K_0 \rightarrow J^3(\tilde{X}_0)$  is 0. Let  $k_{10}, k_{20} \in K_0$  and  $k_1, k_2$  be their representatives in  $K$ . Let us find a cycle  $D$  on  $\tilde{X}_0$  which is the image of  $k_{10} - k_{20}$ :

$$D = \tilde{\chi}(k_1) + \tilde{\chi}(i_K(k_1)) - \tilde{\chi}(k_2) - \tilde{\chi}(i_K(k_2)).$$

We have  $\forall f \in F \quad \Phi(\tilde{\chi}(f)) + \Phi(\tilde{\chi}(i_F(f))) = \text{const}$  and  $i_F(i_K(k_1)) = \alpha(k_1)$ , hence

$$D = \tilde{\chi}(k_1) - \tilde{\chi}(\alpha(k_1)) - \tilde{\chi}(k_2) + \tilde{\chi}(\alpha(k_2)) = U(k_1) - U(\alpha(k_1)) + U(\alpha(k_2)) - U(k_2)$$

i.e.  $D \equiv 0$ .

So, we have proved that  $h^{1,0}(F(X)) = 10$ .

**Remark.** The above considerations imply that there exists a map  $\text{Pr } K/K_0 \rightarrow J^3(\tilde{X}_0) \rightarrow \text{Pr } \hat{H}_6/H_6$ . For a generic covering  $p_{H_6}$  existence of this map implies that  $p_K = p_{H_6}$  which implies in its turn that  $p_K(X_0) = p_{H_6}(X_0)$  is true for any  $X_0$ .

Calculation of  $h^{1,0}(F_0(X))$ , as well as a proof of connectedness of  $F(X)$  and of  $F_0(X)$  is made by the same method, but easier. Analogously to a fibration  $f_F : F_3 \rightarrow \Delta$  we construct a fibration  $f_{0F} : F_{03} \rightarrow \Delta$  having a generic fibre  $f_{0F}^{-1}(t_0) = F_0(X)$  and a special fibre  $f_{0F}^{-1}(0) = F_0(0) = F_0 \cup \bar{R}_0$ , where  $F_0 = \widetilde{F_0(X_0)}$  and  $\bar{R}_0$  is a two-sheeted covering of  $R_0$  ( $R_0$  is a fibration over  $K_0$  whose fibre is  $P^1$ ) ramified at  $K \subset R_0$ , and  $F_0 \cap \bar{R}_0 = K$ . There exists a weight filtration on  $H^1(F_0(0))$ :

$$0 \subset W_0 \subset W_1 = H^1(F_0(0))$$

where  $W_0 = H^1(\Pi(F_0(0)))$ ,  $W_1/W_0 = \text{Ker } H^1(F_0) \oplus H^1(R_0) \rightarrow H^1(K)$ . Since  $\Pi(F_0(0))$  is a segment, then  $W_0 = 0$ . As we have seen (Lemma 6.9),  $H^1(F_0) = 0$ . Since the cohomology map  $H^1(K_0) \rightarrow H^1(\bar{R}_0)$  of the fibration  $\bar{R}_0 \rightarrow K_0$  is an isomorphism, we have:

$$\text{Ker } H^1(\bar{R}_0) \rightarrow H^1(K) = \text{Ker } H^1(K_0) \rightarrow H^1(K) = 0$$

So,  $H^1(F_0(0)) = 0$ . It follows immediately from the Clemens - Schmidt exact sequence of the fibration  $f_{0F}$  that  $H^1(F_0(X)) = 0$  as well.

To calculate  $H^0(F(X))$  we consider the even Clemens - Schmidt exact sequence of the fibration  $f_F$ :

$$0 \rightarrow H^0(F(0)) \rightarrow H^0(F(X)) \xrightarrow{N} H^0(F(X))$$

Connectedness of  $F$  implies that  $h^0(F(0)) = 1$  and that the map  $N$  is 0. This implies connectedness of  $F(X)$  and hence of  $F_0(X)$  as well.

**Proposition 6.10.** The Abel - Jacobi map  $\Phi : \text{Alb } F \rightarrow J^3(X)$  is an epimorphism (here  $F = F(X)$ ).

**Proof.** Let  $F_1$  be the family of straight lines on  $X$ ,  $D \subset F_1 \times F_1$  the incidence divisor, i.e.  $(t_1, t_2) \in D \iff t_1 \neq t_2$  and  $t_1 \cap t_2 \neq \emptyset$ ,  $p_j : D \rightarrow F_1$  a projection to the  $j$ -th component ( $j = 1, 2$ ),  $N = \deg p_j$ , i.e. the number of straight lines meeting a given straight line,  $i : D \rightarrow D$  the natural involution (permutation of  $t_1, t_2$ ) and  $p_D : D \rightarrow D_0$  the factorization of  $D$  by  $i$ . Then  $D_0 \xrightarrow{d} F$  is a set of reducible conics. Let us consider cylinder maps in complex homology

- (a) for conics:  $\Phi_F : H_1(F, \mathbb{C}) \rightarrow H_3(X, \mathbb{C})$ ;
- (b) for straight lines:  $\Phi_{F_1} : H_1(F_1, \mathbb{C}) \rightarrow H_3(X, \mathbb{C})$ .

It is clear that for  $\gamma \in H^1(D)$  we have

$$\Phi_F \circ d_* \circ p_{D*}(\gamma) = \Phi_{F_1}(p_{1*}(\gamma)) + \Phi_{F_1}(p_{2*}(\gamma)) \stackrel{\text{def}}{=} B(\gamma)$$

If  $\gamma = p_1^*(\gamma')$ , where  $\gamma' = H_1(F_1)$ , then

$$B(\gamma) = \Phi_{F_1}(p_{1*}(p_1^*(\gamma'))) + \Phi_{F_1}(p_{2*}(p_1^*(\gamma'))) = N \cdot \Phi_{F_1}(\gamma') + \Phi_{F_1}(p_{1*}(i^*(p_1^*(\gamma'))))$$

According ([20], Section 4.3, Lemma 6) we have

$$\Phi_{F_1}(p_{1*}(i^*(p_1^*(\gamma')))) = (1 - n)\Phi_{F_1}(\gamma')$$

where  $n$  is a number of straight lines passing through a generic point of  $X'_4$ , so  $B(\gamma) = (N + 1 - n)\Phi_{F_1}(\gamma')$ . For  $X$  we have  $N = 11$ ,  $n = 6$ . According to a lemma of Clemens ([20], Section 4.3)  $\Phi_{F_1}$  is an epimorphism, hence  $\gamma' \mapsto B(\gamma) = \Phi_F \circ d_* \circ p_D(\gamma)$  is an epimorphism as well. This implies that  $\Phi_F$  is also an epimorphism, which is equivalent to (6.10).  $\square$

Since  $\text{Alb } F$  and  $J^3(X)$  have the same dimension, we get

**Theorem 6.11.** The Abel - Jacobi map  $\Phi : \text{Alb } F \rightarrow J^3(X)$  is an isogeny.  $\square$

Now we give a sketch of a proof of the following theorem:

**Theorem 6.12.** The class of  $\Phi(F)$  in  $A(J^3(X))$  is  $\frac{2\Theta^8}{8!}$  where  $\Theta$  is the class of a Poincaré divisor.

**Sketch of proof.** For a degeneration  $f : X_4 \rightarrow \Delta$  (see 6.2) we denote  $f^{-1}(t) = X_t$  and  $F(X_t) = F_t$  for  $t \neq 0$ . There exists a complex analytic fibration  $j : J_{11} \rightarrow \Delta$  whose fibre at  $t \in \Delta^* = \Delta - \{0\}$  is  $j^{-1}(t) = J^3(X_t)$  ([52], Section 2). Its fibre

$j^{-1}(0) = J(0)$  at 0 is a fibration over  $J^3(\tilde{X}_0)$  whose fibre is  $\mathbb{C}^*$  ([52], 4.56). There exists a compactification of  $J(0)$ . It is a fibration  $\mu : \tilde{J}_0 \rightarrow J^3(\tilde{X}_0)$  whose fibre is  $P^1$ . It has 2 sections  $\alpha_0, \alpha_\infty : J^3(\tilde{X}_0) \rightarrow \tilde{J}_0$  which map any point  $t \in J^3(\tilde{X}_0)$  to points  $(0), (\infty) \in P^1 = \mu^{-1}(t)$  such that  $J(0) = \tilde{J}_0 - \text{im } \alpha_0 - \text{im } \alpha_\infty$ . There exists also a compactification of  $J_{11}$  which is a fibration  $j_c : J_{11,c} \rightarrow \Delta$  whose non-singular fibre is the same as the one of  $j$  and whose singular fibre is  $j_c^{-1}(0) = J_0$ . Further, there is a map  $\nu : \tilde{J}_0 \rightarrow J_0$  which is the normalization of  $J_0$ .  $\nu$  identifies (with a shift)  $\text{im } \alpha_0$  and  $\text{im } \alpha_\infty$ , outside these subsets  $\nu$  is an inclusion.

Let us consider now the Abel - Jacobi map. We fix a section  $s_0 : \Delta \rightarrow F_3$  such that  $s_0(0) \in F(X_0) - K$ , and for any section  $s : \Delta \rightarrow F_3$  such that  $s(0) \in F(X_0) - K$  we consider an element  $\Phi_s(t) = \Phi(s(t) - s_0(t)) \in J^3(X_t)$ . They form a section  $\Delta^* \rightarrow J_{11}$  which can be expanded to a section  $\Phi_s : \Delta \rightarrow J_{11}$ . We have:  $\Phi_s(0)$  depends only on  $s(0)$  but not on  $s$  and  $\mu(\Phi_s(0)) = \Phi_{\tilde{X}_0}(s(0) - s_0(0))$  ([52], 4.58) where  $\Phi_{\tilde{X}_0}$  is the Abel - Jacobi map of  $\tilde{X}_0$ . We denote by  $\cup_s \Phi_s(t) = F_\Phi(t) \subset j^{-1}(t)$  the image of  $F_t$  under the Abel - Jacobi map shifted by such a way that  $s_0(t)$  goes to 0. For  $t = 0$  we have an inclusion  $F(X_0) - K \hookrightarrow J(0)$ . There exists a commutative diagram where the above inclusion is the left vertical map:

$$\begin{array}{ccccc} F(X_0) - K & \hookrightarrow & \widetilde{F(X_0)} & \twoheadrightarrow & F(X_0) = F_\Phi(0) \\ \downarrow & & \downarrow & & \downarrow \\ J(0) & \hookrightarrow & \tilde{J}_0 & \xrightarrow{\nu} & J_0 \end{array}$$

(vertical maps are inclusions) and we get a family of surfaces  $(F_\Phi \rightarrow \Delta) \subset (j_c : J_{11,c} \rightarrow \Delta)$  whose fibre at  $t \in \Delta$  is  $F_\Phi(t) = F(X_t)$ . Since

$$\mu(\widetilde{F(X_0)}) = \Phi_{\tilde{X}_0}(\widetilde{F(X_0)}) \subset J^3(\tilde{X}_0), \quad J^3(\tilde{X}_0) = \text{Pr}(\hat{H}_6/H_6) \text{ and } \widetilde{F(X_0)} = S(p_{H_6})$$

then according ([28]) the homology class of  $\Phi_{\tilde{X}_0}(\widetilde{F(X_0)})$  in  $H_4(J^3(\tilde{X}_0), \mathbb{Z})$  is

$$\frac{2\Theta^7}{7!} = 2 \sum_{i < j} \gamma_i \times \delta_i \times \gamma_j \times \delta_j \quad (6.13)$$

where  $\Theta$  is the class of a Poincaré divisor and  $(\gamma_i, \delta_i)$  is a symplectic basis of  $H_1(J^3(\tilde{X}_0), \mathbb{Z})$  with respect to the principal polarization associated with  $\Theta \subset J^3(\tilde{X}_0)$ .

There exists a map of topological spaces  $(0, 0) : J^3(X_t) \rightarrow J_0$  ([25]) such that  $(0, 0)(F_\Phi(t))$  is homologically equivalent to  $F_\Phi(0) \hookrightarrow J_0$ . By analogy with proofs of ([25], Lemma 2.6) and ([31], Theorem 8.27) it is possible to show — using (6.13) — that the class of  $F_\Phi(t) \subset J^3(X_t)$  in  $H_4(J^3(X_t), \mathbb{Z})$  is

$$2 \sum_{i < j} \gamma_i \times \delta_i \times \gamma_j \times \delta_j = \frac{2\Theta^8}{8!} \quad \square$$

## Section 7. Geometric interpretation of the tangent bundle theorem and recovering of $X$ by its Fano surface.

We denote  $V_{10} = H^0(F, \Omega_F)^*$ . Here we impose the following condition on  $F$ :

**7.1.** The natural map  $V_{10}^* = H^0(F_m, \Omega_{F_m}) \rightarrow \Omega_{F_m}(c_\Omega)$  is an epimorphism.

Apparently this condition is true for any  $X$  or at least for a generic  $X$ . The theorem on recovering of  $X$  by its Fano surface  $F(X)$  will be proved only for  $X$  that satisfy (7.1).

Let us define a map  $B_8 : F \rightarrow G(2, V_8)$  as follows:  $B_8(f) = \langle \gamma_1(f), \gamma_2(f) \rangle$ . We denote by  $B_{10} : F_m \rightarrow G(2, V_{10})$  a map associated to  $\Omega_{F_m}$ . (7.1) means that  $B_{10}$  is regular at  $c_\Omega$ . It is clear that  $B_8$  factors through  $p_F : F \rightarrow F_0$ . Since  $H^0(F_0, \Omega_{F_0}) = 0$ , we have:  $B_{10}$  factors through  $p_{F_m} : F_m \rightarrow F_{0m}$ , and the corresponding map  $F_{0m} \rightarrow G(2, V_{10})$  is associated to the sheaf  $\Omega_{F_{0m}} \otimes \sigma_m$ .

**Theorem 7.2.** (Geometric interpretation of the tangent bundle theorem). The map  $B_{10} \circ r : F \rightarrow G(2, V_{10})$  is regular. There exists the only epimorphism  $p_{10,8} : V_{10} \rightarrow V_8$  such that

$$B_8 = G(2, p_{10,8}) \circ B_{10} \circ r \quad (7.3)$$

If (7.1) is satisfied then  $p_{10,8}$  is the projection from the straight line  $B_{10}(c_\Omega)$ .

**Proof.** We denote  $H = H^0(F, \tau_{2,M}^*)^*$  and let  $B_H : F \rightarrow G(2, H)$  be the map defined by  $\tau_{2,M}^*$ . Let us consider the long exact cohomology sequence for (4.51):

$$0 \rightarrow H^* \xrightarrow{i_1} V_{10}^* \rightarrow H^0(2O_{l_1 \cup l_2})$$

It is clear that (7.1) is equivalent to a condition  $H = V_8$ . Since the linear envelope of lines  $\langle \gamma_1(f), \gamma_2(f) \rangle = l_8(f) \subset P(V_8)$  is the whole  $P(V_8)$ , there exists an inclusion  $i_2 : V_8^* = H^0(G(2, V_8), \tau_{2,V_8}^*) \hookrightarrow H^* = H^0(F, \tau_{2,M}^*)$ , and we have

$$B_8 = G(2, i_2^*) \circ B_H \quad (7.4)$$

This implies that  $\forall f \in F$   $B_H$  is regular at  $f$ . For  $f \in F - l$  the restriction of (4.51) to  $f$  is

$$0 \rightarrow \tau_{2,M}^*|_f \rightarrow r^*\Omega_{F_m}|_f \rightarrow 0$$

and there exists a commutative diagram

$$\begin{array}{ccccc} H^* & \xrightarrow{i_1} & V_{10}^* & & \\ \downarrow & & \downarrow & & \\ H^0(\tau_{2,M}^*|_f) & = & \tau_{2,M}^*(f) & \rightarrow & H^0(r^*\Omega_{F_m}|_f) = r^*\Omega_{F_m}(f) \end{array} \quad (7.5)$$

Since  $H^* \rightarrow \tau_{2,M}^*(f)$  is an epimorphism, we have that  $V_{10}^* \rightarrow r^*\Omega_{F_m}(f)$  is also an epimorphism, i.e.  $B_{10}$  is regular at  $F - l$ . If so, there exists a continuation of  $B_{10}$  to  $l$ , i.e.  $B_{10}$  is regular on  $F$ . Commutativity of the diagram dual to (7.5) shows that

$$B_H(f) = G(2, i_1^*) \circ B_{10}(f) \quad (7.6)$$

We denote now  $p_{10,8} = (i_1 \circ i_2)^*$ . (7.4) and (7.6) imply that on  $F - l$  we have  $B_8 = G(2, p_{10,8}) \circ B_{10} \circ r$ , hence this is true on the whole  $F$ .

If (7.1) is true then for  $f \in l$  we have  $B_{10} \circ r(f) = B_{10}(c_\Omega)$  which does not depend on  $f$ . But  $B_8(f)$  for  $f \in l$  are different straight lines having no common points. This observation shows that (7.3) can be satisfied only if  $p_{10,8}$  is the projection from the straight line  $B_{10}(c_\Omega)$ .  $\square$

**Theorem 7.7.** Here we suppose that (7.1) is satisfied for  $F_c$ . Then  $X$  such that  $F_c = F_c(X)$  is defined uniquely up to isomorphism.

**Proof.** Let us consider the map  $F_c \rightarrow G(2, V_{10})$  associated to  $\Omega_{F_c}$ . It factors through  $F_{0m}$  and hence it permits us to recover uniquely the point  $c_\Omega \in F_c$ , the straight line  $l_2 \subset F_c$ , the involution  $i_F$  on  $F$  and the straight line  $B_{10}(c_\Omega) \subset P(V_{10})$ . Since (7.1) implies that  $p_{10,8}$  is the projection from  $B_{10}(c_\Omega)$ , we can recover as well the map  $B_8 : F_0 \rightarrow G(2, V_8)$ . Since the straight line  $l \subset F_0$  can be recovered by  $F_c$  and the straight lines  $l_0(c)$  for  $c \in l$  are contained in  $U^*$  and are tangent to the conic  $c_u^*$ , we have that  $U^*$ ,  $c_u^*$  and the isomorphism  $l \rightarrow c_u^*$  are recovered by  $F_c$ .

**Proposition 7.8.**  $c_\Omega \subset U^*$  can be recovered uniquely by  $F_c$ .

**Proof.** We prove

(a) If  $c \in F_0 - l$  and  $l_8(c) \cap U^* \neq \emptyset$  then  $l_8(c) \cap U^* \in c_\Omega$ .

(b) The quantity of these points  $c \in F_0 - l$  is finite and  $> 4$ .

So, having  $F_c$  it is possible to construct more than 4 points on  $c_\Omega$ , i.e. to recover  $c_\Omega$  itself.

(a) can be proved easily. For  $c \in F_0 - l$  we have:  $X \cap l_8(c) = G_4 \cap l_8(c)$ . Since  $U^* \subset G_4$ , we have  $l_8(c) \cap U^* \in G_4$ , and hence  $l_8(c) \cap U^* \in X$ . But  $c_\Omega = X \cap U^*$ .

Let us define now a curve  $S$  as a set of pairs  $(t, c)$  where  $t \in c_\Omega$ ,  $c \in c_u \cap t_V$ . We define a straight line  $j(t, c)$  as a line lying in  $\alpha$ -plane  $\alpha(c)$  (see Section 3) and which is the tangent of  $\alpha(c) \cap \Omega$  at  $t$ . Lemma 3.20 (whose notations are used here) implies that either  $j(t, c) \subset X$  or  $\{ \text{if for } f_0 \in F_0 - l \quad \widetilde{\phi(f_0)} \in \pi(\widetilde{j(t, c)}) \} \quad t \in l_8(f_0)$ . We define a surface

$$R = \bigcup_{(t, c) \in S} \pi(\widetilde{j(t, c)}) \subset P(V^*)$$

It is clear that  $R \supset l$ . Let  $V_4 \in (R - l) \cap (\phi(F_0 - l))$ ,  $V_4 = \phi(f_0)$ ,  $V_4 \in \pi(\widetilde{j(t, c)})$ . Then either

(i)  $l_8(f_0) \cap c_\Omega = \{t\}$  or

(ii) There exists a straight line on  $X$  passing through  $t$ .

Let us find for how many  $V_4 \in (R - l) \cap (\phi(F_0 - l))$  we have the case (ii). Since the union of all straight lines on  $X$  is an intersection of  $X$  and a hypersurface of degree 10, there are 20 points on  $c_\Omega$  which belong to a straight line on  $X$ . Let  $t$  and  $j(t)$  are one of these points and lines. We define  $c = \cap_{t' \in j(t)} t'_V$ ; it is clear that  $c \in c_u \cap t_V$  and  $j(t) = j(t, c)$ . Since  $j(t)$  meets 11 straight lines on  $X$ , we have: a straight line  $\pi(\widetilde{j(t, c)}) \subset R$  meets  $\phi(F_0)$  at 11 points. But one of them belongs to  $l$ . We get that for  $20 \cdot (11 - 1) = 200$  intersection points of  $R - l$  and  $\phi(F_0 - l)$  there exist corresponding lines on  $X$ .

Let us find now  $\#(R - l) \cap (\phi(F_0 - l))$ . Let  $r : \widetilde{P(V^*)} \rightarrow P(V^*)$  be the blowing up along  $l$ ,  $\tilde{R}$  and  $\tilde{F}_0$  the proper inverse images of  $R$  and  $\phi(F_0)$  respectively. Then  $\#(R - l) \cap (\phi(F_0 - l)) = \langle \tilde{R}, \tilde{F}_0 \rangle_{\widetilde{P(V^*)}}$ .

**Lemma 7.9.**  $\langle \tilde{R}, \tilde{F}_0 \rangle_{\widetilde{P(V^*)}} = 36d + 8$  where  $d = \deg R$ .

**Proof.**  $\widetilde{P(V^*)}$  is a variety of pairs  $(V_2, V_4)$  where  $V_2 \subset V_4 \subset V$  and  $V_2 \subset U_3$ ;  $r(V_2, V_4) = V_4$ . There exists an isomorphism

$$\widetilde{P(V^*)} = P_{U^*}(O \oplus O \oplus O(1)) \xrightarrow{\mu} U^*$$

and  $\mu(V_2, V_4) = V_2 \in U^*$ . The ring  $A(\widetilde{P(V^*)})$  has generators  $D = \mu^*(c_1(O_{U^*}(1)))$ ,  $\tilde{H} = c_1(O_\mu(1))$  and relations  $D^3 = 0$ ,  $\tilde{H}^3 = D\tilde{H}^2$ . Further, we have

$$r^{-1}(l) = \tilde{H} - D, \quad r^{-1}(H) = \tilde{H}, \quad r_*(\tilde{H}^2) = r_*(\tilde{H}D) = r_*(D^2) = H^2$$

where  $H = c_1(O(1))$ . Let us find the class of  $R$  in  $A(\widetilde{P(V^*)})$ . By definition,

$$\{(V_2, V_4) \in \tilde{R} \iff P(V_2) = t_V \text{ for some } t \in c_\Omega \text{ and } P(V_4) \supset \pi(j(t, c))\}$$

where  $c \in t_V \cap c_u$ . So,  $\mu(\tilde{R}) = c_\Omega$ , and for  $t \in c_\Omega$   $\mu^{-1}(t) \cap \tilde{R}$  is a pair of straight lines  $\pi(j(t, c_i))$  ( $i = 1, 2$ ), where  $\{c_1, c_2\} = t_V \cap c_u$ . This means that  $\tilde{R}$  is a divisor in  $\mu^{-1}(c_\Omega)$  and  $O_{\mu^{-1}(c_\Omega)}(\tilde{R}) = L \otimes O_\mu(2)$  where  $L$  is an invertible sheaf on  $c_\Omega$ . Since  $\mu^{-1}(c_\Omega)$  is a divisor on  $\widetilde{P(V^*)}$  of degree  $2D$ , we have:  $\text{cl}(\tilde{R}) = 4D\tilde{H} + nD^2$  where  $n$  is defined by the condition  $r_*(\tilde{R}) = dH^2 = (4 + n)H^2$ , i.e.  $n = d - 4$ .

Let us find now the class of  $\tilde{F}_0$  in  $A(\widetilde{P(V^*)})$ . Let us assume that there exists a locally free sheaf  $E$  of rank 2 on  $P(V^*)$  such that  $\phi(F_0)$  is the set of zeros of a section of  $E$ . Then  $\tilde{F}_0$  is the set of zeros of a section of  $r^*(E) \otimes O_r(1)$  on  $\widetilde{P(V^*)}$  and

$$\text{cl}(\tilde{F}_0) = c_2(r^*(E) \otimes O_r(1)) = r^*c_2(E) + r^*c_1(E) \cdot c_1(O_r(1)) + c_1(O_r(1))^2.$$

We have  $c_2(E) = \deg \phi(F_0) = 39H^2$ ,  $c_1(O_r(1)) = D - \tilde{H}$ , and  $c_1(E)$  can be easily found by consideration of the exact sequence of normal sheaves for inclusions  $l \hookrightarrow \phi(F_0) \hookrightarrow P(V^*)$ :

$$0 \rightarrow O_l(-1) \rightarrow 3O_l(1) \rightarrow E|_l \rightarrow 0$$

Namely,  $\det E|_l = O_l(4)$ , i.e.  $c_1(E) = 4H$  and

$$\text{cl} \tilde{F}_0 = 39\tilde{H}^2 + 4\tilde{H}(D - \tilde{H}) + (D - \tilde{H})^2 = 36\tilde{H}^2 + 2\tilde{H}D + D^2$$

Really,  $E$  does not exist, but it is clear that the above formalism is correct. So,  $\langle \tilde{R}, \tilde{F}_0 \rangle_{\widetilde{P(V^*)}} = (4\tilde{H}D + (d - 4)D^2)(36\tilde{H}^2 + 2\tilde{H}D + D^2) = 36d + 8$ .  $\square$

This means that there are  $36d + 8 - 200$  straight lines of type  $l_8(f_0)$  which meet  $c_\Omega$ . Since this number is  $\geq 0$ , we get that  $d \geq 6$  and this number is  $\geq 24$ . So, we can recover  $c_\Omega$  uniquely by  $F_c$ .  $\square$

Let us consider now the birational isomorphism  $b : G_4 \rightarrow P^4$  (see (3.4)) which is the restriction to  $G_4$  of the projection  $p_{85} : P(V_8) \rightarrow P^4 = P(V_8/U_3^*)$ .

**Proposition 7.10.** It is possible to recover the normcubic  $c_3 \subset P^3 \subset P^4$  and the isomorphism  $\psi_2 : c_u \rightarrow c_3$  by  $F_c$ .

**Proof.** For  $f_0 \in F_0$  we let  $l_5(f_0) = G(2, p_{85})(l_8(f_0))$ , i.e.  $l_5$  is a map from  $F_0$  to  $G(2, V_8/U_3^*)$ . Since  $l$  is a divisor of  $F_0$ , then  $l_5(f_0)$  is defined also for a generic point  $f_0 \in l$ . We shall consider  $l_5(f_0)$  also as a straight line in  $P^4$ . A natural map of straight lines  $p_{85}|_{l_8(f_0)} : l_8(f_0) \rightarrow l_5(f_0)$  is defined also for a generic point  $f_0 \in l$ . Analogously, the map  $\xi_5 = p_{85} \circ \xi : P_{F_0}(\tau_{2,M}) \rightarrow P^4$  is defined at a generic point of  $P_l(\tau_{2,M})$ .

It is clear that in  $P_{F_0}(\tau_{2,M})$  we have:  $S = \bar{W} \cap P_l(\tau_{2,M})$  (here we identify  $c_u$  and  $l$  via  $\psi$ ). Let us consider a commutative diagram

$$\begin{array}{ccccccc} S & \hookrightarrow & \bar{W} & \xrightarrow{\xi|_W} & W & \hookrightarrow & X \hookrightarrow G_4 \\ \downarrow & & \downarrow & & & & \downarrow \\ P_l(\tau_{2,M}) & \hookrightarrow & P_{F_0}(\tau_{2,M}) & \xrightarrow{\xi_5} & & & P^4 \end{array}$$

where the left and middle vertical maps are inclusions, the right vertical map is  $b$ . Let  $t_1 \in c_u$ , then  $l_8(\psi(t_1)) \subset U^*$ , and let  $\{t', t''\} = l_8(\psi(t_1)) \cap c_\Omega$ . We denote  $\{t_1, t_2\} = t'_V \cap c_u$  and  $\{t_1, t_3\} = t''_V \cap c_u$ . Then  $(t', t_1) \in S$ ,  $(t', t_2) \in S$ . The image of these elements in  $G_4$  is  $t'$ , hence their images in  $P^4$  belong to a bisecant  $\langle \psi_2(t_1), \psi_2(t_2) \rangle$  of  $c_3$ . Since  $c_\Omega$  can be recovered by  $F_c$ , we have: for  $t_1 \in c_u$  we can find images of points  $(t', t_1)$  and  $(t', t_2)$  in  $P^4$  going along the left-lower way of the above diagram, and hence we can find the bisecant  $\langle \psi_2(t_1), \psi_2(t_2) \rangle$ . Analogously we can find the bisecant  $\langle \psi_2(t_1), \psi_2(t_3) \rangle$ ; their intersection point indicates us the point  $\psi_2(t_1) \in P^4$ .  $\square$

Let us consider an isomorphism  $I : P(V_8) \rightarrow P(V'_8)$  and a map  $b^{-1} : P^4 \rightarrow I(G_4)$  such that  $b^{-1} \circ b = I|_{G_4}$ . Since we can recover uniquely  $c_3 \subset P^4$ , we can also recover  $b^{-1} : P^4 \rightarrow I(G_4)$  and  $P(V'_8)$  as the linear envelope of  $I(G_4)$ .

**Proposition 7.11.** Isomorphism  $I : P(V_8) \rightarrow P(V'_8)$  can be recovered by  $F_c$ .

**Proof.** Let  $\phi(f_0) = V_4$  where  $f_0 \in F_0 - l$ , and let  $\{t\} = Q_G(V_4) \cap U^*$ . Then (see 3.4)  $b(Q_G(V_4)) = P^2$  and since points  $b(\gamma_1(f_0))$ ,  $b(\gamma_2(f_0))$  belong to  $P^2$ , we have:  $l_5(f_0) \subset P^2$ . This implies that  $b^{-1}(l_5(f_0))$  is a conic on  $I(Q_G(V_4))$  passing through  $I(t)$ ,  $I(\gamma_1(f_0))$ ,  $I(\gamma_2(f_0))$ . Since  $b$  is the restriction of  $p_{85} : P(V_8) \rightarrow P^4$  to  $G_4$ , we have that for  $a \in l_8(f_0)$  points  $I(t)$ ,  $I(a)$  and  $b^{-1}(p_{85}(a))$  are collinear.

We can recover an isomorphism  $S^2(c_3) \rightarrow I(U^*)$  and hence  $I|_{U^*} : U^* \rightarrow I(U^*)$  as well. For  $f_0$  we can recover a point  $l_5(f_0) \cap P^3$ . There exists only one bisecant of  $c_3$  passing through this point, and the linear envelope of this bisecant and  $l_5(f_0)$  is equal clearly  $P^2 = b(Q_G(V_4))$ . So, we can recover this  $P^2$ .

Let now  $f_1, f_2 \in F_0 - l$  be such that  $l_5(f_1) \cap l_5(f_2) = \{m\}$ , where  $m = m(f_1, f_2) \in P^4$ , and  $m_1 \in l_8(f_1)$ ,  $m_2 \in l_8(f_2)$  are inverse images of  $m$ :  $p_{85}(m_1) = p_{85}(m_2) = m$ . Let, further,  $t_1 = Q_G(\phi(f_1)) \cap U^*$ ,  $t_2 = Q_G(\phi(f_2)) \cap U^*$ . Then the point  $b^{-1}(m)$  belongs to both lines  $\langle I(t_i), I(m_i) \rangle$  ( $i = 1, 2$ ), hence  $I^{-1}(b^{-1}(m)) = \langle t_1, m_1 \rangle \cap \langle t_2, m_2 \rangle$ , because these lines are different. If  $f_1, f_2$  are given, then we can construct points  $m, m_1, m_2, t_1, t_2, b^{-1}(m)$ , and hence  $I^{-1}(b^{-1}(m))$ . So, there are many points in  $P(V_8)$  whose  $I$ -image can be constructed. To prove that they permit to recover uniquely the linear map  $I$  it is sufficient to prove that all points of type  $b^{-1}(m(f_1, f_2))$  do not belong to a hyperplane containing  $I(U^*)$ , or — the same — to prove that all points of type  $m(f_1, f_2)$  do not belong to a hyperplane in  $P^4$ .

**Lemma 7.12.** Let  $Y$  be a surface on  $G(2, V)$ ,  $d = c_1(\tau_{2,V}|_Y)^2 - c_2(\tau_{2,V}|_Y)$ . Then for a generic point  $y \in Y$  we have: the straight line  $y_V$  meets  $d - 3$  straight lines of type  $y'_V$  where  $y' \in Y - y$ .

**Proof.** Follows easily from a calculation of intersection index on  $\tilde{G}_y$ .  $\square$

Chern classes of  $l_5^*(\tau_2)$  (this sheaf corresponds to the inclusion  $l_5 \times F_0 \rightarrow G(2, (V_8/V_3)^*)$ ) can be easily found by using exact sequences (4.51) and

$$0 \rightarrow \tau_2^*|_{F_0} \rightarrow \tau_{2,M}^*|_{F_0} \rightarrow i_*(\tau_{2,M}^*|_l) \rightarrow 0$$

where  $i : l \rightarrow F_0$ . We get:  $d = 168$ . This means that straight lines of type  $l_5(f_0)$  for  $f_0 \in F$  do not belong to one hypersurface and a generic straight line of this type meets 165 others. This implies that all points of type  $m(f_1, f_2)$  do not belong to a hyperplane in  $P^4$ .  $\square$

Now we can recover  $G_4 \subset P(V_8)$  by  $F_c$ , because  $G_4 = I^{-1}(b^{-1}(P^4))$ . It is clear that for  $f_0 \in F_0 - l$

$$\{\gamma_1(f_0), \gamma_2(f_0)\} = G_4 \cap l_8(f_0)$$

hence we can recover  $W$  by  $F_c$ . Since  $O_X(W) = O_X(21)$  and  $X$  is an intersection of quadrics in  $P(V_8)$ ,  $X$  is an intersection of all quadrics in  $P(V_8)$  which contain  $W$ . So, we can recover uniquely  $X$  by  $F_c$ , Q.E.D.  $\square$

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